Homework 11 Morally Due May 5

1. (0 points) Where and when is the final?

2. (40 points) For each of the following problems either find the $a, b$ requested OR show that its NOT POSSIBLE. You must prove your answer in all cases.

   (a) Find an $a, b$ with $a < b$ such that the following holds: For all 2-colorings of the $a \times b$ grid there exists at least TWO mono rectangles. (They do not have to be the same color but they could be.)

   (b) Find an $a, b$ with $a < b$ such that the following holds: For all 2-colorings of the $a \times b$ grid there exists at least TWO mono rectangles OF THE SAME COLOR.

   (c) Find an $a, b$ with $a < b$ such that the following holds: For all 2-colorings of the $a \times b$ grid there exists at least TWO mono rectangles OF DIFFERENT COLORS.

   **SOLUTION TO PROBLEM TWO**

   a)

   ANSWER ONE:

   Let $a = 4$. We will determine $b$ later. Any 2-coloring of a column leads to either

   (a) Some color occuring $\geq 3$ times, or

   (b) Two colors that both occur twice.

   If two columns are colored the same then one of the following occurs:

   - The columns are of type 1 above. Then it will look like this (or variants that are essentially the same).

$$\begin{matrix} \cdots & R & \cdots & R & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & B & \cdots & B & \cdots \end{matrix}$$

     Note that there are actually 3 mono rectangles.

- The columns are of type 2 above. Then it will look like this (or variants that are essentially the same).

$$\begin{array}{ccccc} \cdots & R & \cdots & R & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & B & \cdots & B & \cdots \\ \cdots & B & \cdots & B & \cdots \end{array}$$

  Note that there are 2 mono rectangles.

We take $b = 2^4 + 1 = 17$. This guarantees a repeat column color and hence at least 2 mono rectangles.

ANSWER TWO Recall that no matter how you 2-color a $3 \times 7$ there will be a mono rectangle. So no matter how you color a $3 \times 14$ grid there will be two mono rectangles.

b)

ANSWER ONE:

If $a = 5$ and $b = 2^5 + 1 = 33$ then we will have two columns that are the same color. The column itself will have three of the same color. So we get something like this:

$$\begin{array}{ccccc} \cdots & R & \cdots & R & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & R & \cdots & R & \cdots \\ \cdots & B & \cdots & B & \cdots \\ \cdots & B & \cdots & B & \cdots \end{array}$$

This has three rectangles OF THE SAME COLOR.

ANSWER TWO: Recall that no matter how you 2-color a $3 \times 7$ there will be a mono rectangle. So no matter how you color a $3 \times 21$ grid there will be three mono rectangles. Two of them have to be the same color.

c) This is not possible. Assume that $a, b$ satisfy this property. Color $a \times b$ ALL RED. There will not be two DIFFERENT colored rectangles.

3. (30 points)

(a) Show that there is no 1-1, onto, ORDER-PRESERVING function $f$ from $\mathsf{N}$ to $\mathsf{Q}$. (A function is ORDER-PRESERVING if $x < y$ implies $f(x) < f(y)$.

(b) Show that there is no 1-1, onto, ORDER-PRESERVING (defined later) function $f$ from $\mathsf{Q}$ to $\mathsf{N}$. (A function is ORDER-PRESERVING if $x < y$ implies $f(x) < f(y)$.

**SOLUTION TO PROBLEM THREE**

a) Assume, by way of contradiction, that $f$ is an order preserving bijection from $\mathsf{N}$ to $\mathsf{Q}$.

Let $f(0) = \alpha$ and $f(1) = \beta$. Let $\gamma = \frac{\alpha+\beta}{2}$. Note that $\alpha < \gamma < \beta$.

Since $f$ is onto there exists a natural number $a$ that maps to $\gamma$. Since $\alpha < \gamma < \beta$. and the function $f$ is order preserving, the number $a$ must be a natural number strictly between 0 and 1. There is no such number. Contradiction.

b) If there was an order-pres bijection from $\mathsf{Q}$ to $\mathsf{N}$ then its inverse is an order-pres bijection from $\mathsf{N}$ to $\mathsf{Q}$. By part $a$ this cannot exist.

**END OF SOLUTION TO PROBLEM THREE**

4. (30 points) For each of the following relations $R$, answer and prove all of the following questions: (1) is $R$ reflexive? (2) is $R$ symmetric? (3) is $R$ transitive?

   You must PROVE all of your assertions.

   (a) Relation is over the set $\mathsf{R}$. $R(x, y)$ iff $|x - y| \geq 1$.

   (b) Relation is over the set $\mathsf{R}$. $R(x, y)$ iff $x - y \in \mathsf{Q}$.

   (c) Relation is over the set $\mathsf{Z}$. $R(x, y)$ iff $x$ divides $y$.

**SOLUTION TO PROBLEM FOUR**

a)

$R$ is NOT reflexive: $|1 - 1| = 0 < 1$.

$R$ is symmetric: If $|x - y| \geq 1$ then $|y - x| = |x - y| \geq 1$.

$R$ is NOT Transitive. Let $x = 2$, $y = 3$, $z = 2$.

$R(x, y)$ holds, $R(y, z)$ holds, but $R(x, z)$ does not hold.

If you want an example where all of the numbers are distinct then use $x = 2$, $y = 3$, $z = 1.9$.

b)

$R$ is reflexive: $x - x = 0$, a rational.

$R$ is symmetric: if $x - y = q$, a rational, then $y - x = -q$, a rational.

$R$ is transitive:

$x - y = q_1$, a rational

$y - z = q_2$, a rational

Add these to get

$x - z = q_1 + q_2$, a rational.

c)

$R$ is reflexive: $x$ always divides $x$.

$R$ is NOT symmetric: 2 divides 4 but 4 does not divide 2.

$R$ is transitive:

Assume $x$ divides $y$ and $y$ divides $z$. Since $x$ divides $y$ there exists $a$, $y = ax$. Since $y$ divides $z$ there exists $b$, $z = by$.

So $z = by = bax$, so $x$ divides $z$.

**END OF SOLUTION TO PROBLEM FOUR**