Theorem

Theorem

$(\forall n \geq 3)(\exists d_1 < \cdots < d_n)[1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n}]$.

We prove this four different ways by induction.

Prove $P(3)$

Proof I: The Idea

Use

$$\frac{1}{d} = \frac{1}{d+1} + \frac{1}{d(d+1)}.$$

Will help get $P(n-1) \to P(n)$

Proof I: Formally

Base Case: $P(3)$ via $(2, 3, 6)$ since $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.
IH: $P(n-1)$: $(\exists d_1 < \cdots < d_{n-1})[1 = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}}]$.
IS: Since $\frac{1}{d_{n-1}} = \frac{1}{d_{n-1}+1} + \frac{1}{d_{n-1}(d_{n-1}+1)}$

$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-2}} + \frac{1}{d_{n-1}} =$
$\frac{1}{d_1} + \cdots + \frac{1}{d_{n-2}} + \frac{1}{d_{n-1}+1} + \frac{1}{d_{n-1}(d_{n-1}+1)}$.

Note that $d_{n-2} < d_{n-1} < d_{n-1} + 1 < d_{n-1}(d_{n-1} + 1)$.

Proof II: The Idea

Use

$$\frac{1}{d} = \frac{1}{2d} + \frac{1}{3d} + \frac{1}{6d}.$$

Will help get $P(n-2) \rightarrow P(n)$

Proof II: Formally

Base Case:

$P(3)$ via $(2,3,6)$ since $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$.

$P(4)$ via $(2,3,7,42)$ since $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=1$.

IH: $P(n-2)$: $(\exists d_1 < \cdots < d_{n-2})[1 = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-2}}]$.

IS: Since $\frac{1}{d_{n-2}} = \frac{1}{2d_{n-2}} + \frac{1}{3d_{n-2}} + \frac{1}{6d_{n-2}}$

$$1 = \frac{1}{d_1}+\cdots+\frac{1}{d_{n-3}}+\frac{1}{d_{n-2}} = \frac{1}{d_1}+\cdots+\frac{1}{d_{n-3}}+\frac{1}{2d_{n-2}}+\frac{1}{3d_{n-2}}+\frac{1}{6d_{n-2}}.$$

Note that $d_{n-3} < d_{n-2} < 2d_{n-2} < 3d_{n-2} < 6d_{n-2}$.

Variant on Induction

What if you have $P(n-2) \to P(n)$?
Let $a \in N$.

$$\begin{array}{ll} & P(a) \\ & P(a+1) \\ & (\forall n \geq a+2)[P(n-2) \to P(n)] \\ \therefore & (\forall n \geq a)(P(n)) \end{array}$$

(Can generalize.)

Why Does It Work?

$$\begin{array}{rl} & P(a) \\ & P(a+1) \\ & (\forall n \geq a+2)[P(n-2) \wedge P(n)] \\ \therefore & (\forall n \geq a)(P(n)) \end{array}$$

1) Have $P(a)$, $P(a+1)$
2) From $P(a)$, $P(a) \rightarrow P(a+2)$ get $P(a+2)$
3) From $P(a+1)$, $P(a+1) \rightarrow P(a+3)$ get $P(a+3)$
4) From $P(a+2)$, $P(a+2) \rightarrow P(a+4)$ get $P(a+4)$
5) From $P(a+3)$, $P(a+3) \rightarrow P(a+5)$ get $P(a+5)$.
etc.

Proof III: Idea

Inductively assume that the largest denominator d is even.
Use

$$\frac{1}{d} = \frac{3}{3d} = \frac{2}{3d} + \frac{1}{3d} = \frac{1}{3d/2} + \frac{1}{3d}.$$

Will help get $P(n-1) \rightarrow P(n)$.
But need that d is even! We abbreviate this $d \equiv 0$.

Proof III: Formally

Strengthen the induction hypothesis:

Theorem

$(\forall n \geq 3)(\exists d_1 < \cdots < d_n)[d_n \equiv 0 \wedge 1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n}]$.

Base Case: $n = 3 : (2, 3, 6)$—$6 \equiv 0 \wedge \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.

IH: $P(n-1)$:

$(\exists d_1 < \cdots < d_{n-1})[d_n \equiv 0 \wedge 1 = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} = 1]$.

IS: Since $\frac{1}{d_{n-1}} = \frac{1}{3d_{n-1}/2} + \frac{1}{3d_{n-1}}$.

$$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-2}} + \frac{1}{d_{n-1}} = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-2}} + \frac{1}{3d_{n-1}/2} + \frac{1}{3d_{n-1}} = 1$$

Note that $d_{n-2} < d_{n-1} < 3d_{n-1}/2 < 3d_{n-1}$ and that $3d_{n-1} \equiv 0$.

Strengthening the IH

You want to prove $P(n)$ by induction.

$P(n-1) \implies P(n)$ seems hard to do.

BUT

$P(n-1) \wedge Q(n-1) \rightarrow P(n) \wedge Q(n)$ is easy to do.

So prove

Base Case: $P(1) \wedge Q(1)$

IS: $(\forall n \geq 2)[P(n-1) \wedge Q(n-1) \rightarrow P(n) \wedge Q(n)]$.

Thoughts on Strengthening the IH

You get to **assume** more ($P(n-1) \wedge Q(n-1)$ instead of just $P(n-1)$. Yeah!

You have to prove more ($P(n) \wedge Q(n)$ instead of $P(n)$.) Boo!

Sometimes its easier to prove a harder theorem

Discuss

Ioana's Proof

Bill: Note that $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$. I know three proofs that $(\forall n \geq 3)(\exists d_1 < \cdots < d_n)[1 = \frac{1}{d_1}+\cdots+\frac{1}{d_n}=1]$.
Try to proof it and I'll be curious which one of mine you come up with.

Ioana: Hmmm, you would not have shown me $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$ unless it was used in the proof.

Bill: Oh. Well. Uh ... (worried she may go down the wrong track)

Ioana's Proof Idea

If have

$$1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n} = 1$$

then

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}\left(\frac{1}{d_1} + \cdots + \frac{1}{d_n}\right).$$

Can use to show $P(n) \rightarrow P(n+2)$. Need $P(3)$ and $P(4)$ as base cases. We omit details.