Outline

We will cover (over the next few weeks)

- Induction
- Strong Induction
- Constructive Induction
- Structural Induction

Induction

$$
\begin{array}{ll}
& P(1) \\
& (\forall n \geq 2)[P(n-1) \rightarrow P(n)] \\
\therefore & (\forall n \geq 1)[P(n)]
\end{array}
$$

Why Does This Work? I

$$P(1)$$
$$(\forall n \geq 2)[P(n-1) \rightarrow P(n)]$$

1) Have $P(1)$
2) Have $P(1) \rightarrow P(2)$
3) From $P(1)$ and $P(1) \rightarrow P(2)$ have $P(2)$.
4) Have $P(2) \rightarrow P(3)$.
5) From $P(2)$ and $P(2) \rightarrow P(3)$ have $P(3)$.
etc.

Why Does This Work? II

$$P(1)$$
$$(\forall n \geq 2)[P(n-1) \rightarrow P(n)]$$

For any **particular** n we can construct a proof of $P(n)$.
Induction proves that, for all n, there is a proof of $P(n)$.
Hence $(\forall n)[P(n)]$.

Requirements

Mathematical Inductive proofs must have:

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- Base case: $P(1)$
  Usually easy

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- Inductive hypothesis: Assume $P(n-1)$
- Inductive step: Prove $P(n-1) \rightarrow P(n)$

Arithmetic series: A first example

Example

For all $n \geq 1$

$$\begin{aligned}\sum_{i=1}^{n} 4i - 2 &= 2 + 6 + 10 + \ldots + (4n-6) + (4n-2) \\ &= 2n^2\end{aligned}$$

Do in class.

Arithmetic series: Gauss's sum

Example

For all $n \geq 1$

$$\begin{aligned}\sum_{i=1}^{n} i &= 1 + 2 + 3 + \ldots + (n-1) + n \\ &= \frac{n(n+1)}{2}\end{aligned}$$

Do in class.

Sum of powers of 2

Example

For all $n \geq 1$

$$\sum_{k=0}^{n-1} 2^k = 1 + 2 + 4 + 8 + \ldots + 2^{n-2} + 2^{n-1}$$
$$= 2^n - 1$$

Do in class.

Geometric Series

Example

For all $n \geq 1$ and real $r \neq 1$

$$\begin{aligned}\sum_{k=0}^{n-1} r^k &= 1 + r + r^2 + r^3 + \ldots + r^{n-2} + r^{n-1} \\ &= \frac{r^n - 1}{r - 1}\end{aligned}$$

Do in class.

A divisibility property

Example

For all integers $n \geq 0$

$$n^3 \equiv n \pmod 3$$

Do in class.

Use $P(n-1) \rightarrow P(n)$ and/or start with Induction Hypothesis.

A recurrence relation

Example

Let

$$a_n = \begin{cases} a_{n-1} + (2n-1) & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

Then

$$a_n = n^2 \qquad \text{for } n \geq 1$$

Do in class.

An inequality

Example

For all integers $n \geq 3$

$$2n + 1 < 2^n$$

Do in class.

Another inequality

Example

For all integers $n \geq 0$ and real $x \geq 0$

$$1 + nx \leq (1 + x)^n$$

Do in class.

Catalan Numbers

Example

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

For all integers $n \geq 1$

$$\frac{(2n)!}{n!(n+1)!} \geq \frac{4^n}{(n+1)^2}$$

Do in class.

Also on handout.

A less mathematical example

Example

If all we have is 2 cent and 5 cent coins, we can make change for any amount of money at least 4 cents.

Do in class.

A recurrence relation

Example

Start with $a_0 = 1$. $a_1 = a_0 + 1 = 1 + 1 = 2$.
$a_2 = (a_0 + a_1) + 1 = (1 + 2) + 1 = 4$.
$a_3 = (a_0 + a_1 + a_2) + 1 = (1 + 2 + 4) + 1 = 8$.
$a_4 = (a_0 + a_1 + a_2 + a_3) + 1 = (1 + 2 + 4 + 8) + 1 = 16$.

In general,

$$\begin{aligned} a_n &= \left(\sum_{i=0}^{n-1} a_i\right) + 1 = (a_0 + a_1 + a_2 + \ldots + a_{n-1}) + 1 \\ &= 2^n \end{aligned}$$

A recurrence relation

Example

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In general,

$$\begin{aligned} a_n &= \left(\sum_{i=0}^{n-1} a_i\right) + 1 = (a_0 + a_1 + a_2 + \ldots + a_{n-1}) + 1 \\ &= 2^n \end{aligned}$$

Proof on next slide!!!

Proof of recurrence relation by mathematical induction

Theorem

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ \left(\sum_{i=0}^{n-1} a_i\right) + 1 \;=\; a_0 + a_1 + \ldots + a_{n-1} + 1 & \text{if } n \geq 1 \end{cases}$$

Then $a_n \;=\; 2^n$.

Proof by Mathematical Induction. Base case easy.

Induction Hypothesis: Assume $a_{n-1} \;=\; 2^{n-1}$.
Induction Step:

$$a_n \;=\; \left(\sum_{i=0}^{n-1} a_i\right) + 1 \;=\; \left(\sum_{i=0}^{n-2} a_i\right) + a_{n-1} + 1 \qquad \text{Now what?}$$

Proof of recurrence relation by mathematical induction

Theorem

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ \left(\sum_{i=0}^{n-1} a_i\right) + 1 \;=\; a_0 + a_1 + \ldots + a_{n-1} + 1 & \text{if } n \geq 1 \end{cases}$$

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Proof by Mathematical Induction. Base case easy.

Induction Hypothesis: Assume $a_{n-1} \;=\; 2^{n-1}$.
Induction Step:

$$\begin{aligned} a_n &= \left(\sum_{i=0}^{n-1} a_i\right) + 1 \;=\; \left(\sum_{i=0}^{n-2} a_i\right) + a_{n-1} + 1 \quad \color{red}{\text{Now what?}} \\ &= (a_{n-1} - 1) + a_{n-1} + 1 \;=\; 2a_{n-1} \;=\; 2 \cdot 2^{n-1} \;=\; 2^n \;. \end{aligned}$$

$\square$

Strong Induction

Recall weak mathematical induction:

$$\begin{array}{rl} & P(1) \\ & (\forall n \geq 2)[P(n-1) \rightarrow P(n)] \\ \therefore & (\forall n \geq 1)[P(n)] \end{array}$$

Strong Induction

Recall weak mathematical induction:

$$\begin{array}{rl} & P(1) \\ & (\forall n \geq 2)[P(n-1) \rightarrow P(n)] \\ \therefore & (\forall n \geq 1)[P(n)] \end{array}$$

Strong induction:

$$\begin{array}{rl} & P(1) \\ & (\forall n \geq 2)[P(1) \wedge P(2) \wedge \cdots \wedge P(n-1) \rightarrow P(n)] \\ \therefore & (\forall n \geq 1)[P(n)] \end{array}$$

Why Does Strong Induction Work?

$$P(1)$$
$$(\forall n \geq 2)[P(1) \wedge P(2) \wedge \cdots \wedge P(n-1) \rightarrow P(n)]$$

1) Have $P(1)$
2) Have $P(1) \rightarrow P(2)$
3) From $P(1)$ and $P(1) \rightarrow P(2)$ have $P(2)$.
4) Have $P(1) \wedge P(2) \rightarrow P(3)$.
5) From $P(1), P(2))$ and $P(1) \wedge P(2) \rightarrow P(3)$ have $P(3)$.
etc.

Proof of recurrence relation by strong induction

Theorem

$$a_n = \begin{cases} 1 & \textit{if } n = 0 \\ \left(\sum_{i=0}^{n-1} a_i\right) + 1 \ = \ a_0 + a_1 + \ldots + a_{n-1} + 1 & \textit{if } n \geq 1 \end{cases}$$

Then $a_n \ = \ 2^n$.

Proof by Strong Induction. Base case easy.

Induction Hypothesis: Assume $a_i \ = \ 2^i$ for $0 \leq i < n$.
Induction Step:

$$\begin{aligned} a_n &= \left(\sum_{i=0}^{n-1} a_i\right) + 1 \ = \ \left(\sum_{i=0}^{n-1} 2^i\right) + 1 \\ &= (2^n - 1) + 1 \ = \ 2^n \ . \end{aligned}$$

$\square$

Another recurrence relation

Example

Let

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} - a_{n-3} & \text{if } n \geq 3 \end{cases}$$

Then a_n is odd, for $n \geq 0$.

Do in class.

Yet another recurrence relation

Example

Let

$$a_i = \begin{cases} 0 & \text{if } i = 0 \\ 7 & \text{if } i = 1 \\ 2a_{i-1} + 3a_{i-2} & \text{if } i \geq 2 \end{cases}$$

Then for all integers $i \geq 0$

$$a_i \equiv 0 \pmod{7}$$

Do in class.

And yet another recurrence relation

Example

Let

$$a_i = \begin{cases} 0 & \text{if } i = 1 \\ 2 & \text{if } i = 2 \\ 3a_{\lfloor \frac{i}{2} \rfloor} & \text{if } i \geq 3 \end{cases}$$

Then for all integers $i \geq 1$, a_i is even.

Do in class.

Size of prime numbers

Example

Let p_n be the nth prime number. Then

$$p_n \leq 2^{2^n}$$

Proof in class.

Context Free Grammars (CFGs)

We formalize later. EXAMPLE of **CFG**

G:

$S \to ASB$

$S \to e$

$A \to a$

$B \to b$

What is $L(G)$ (strings generated)?

Context Free Grammars (CFGs)

We formalize later. EXAMPLE of **CFG**

G:

$S \to ASB$

$S \to e$

$A \to a$

$B \to b$

What is $L(G)$ (strings generated)?

$$\{a^n b^n : n \in \mathrm{N}\}.$$

Context Free Grammars (CFGs)

Def: A **CFG** is $G = (N, \Sigma, R, S)$ where

- N is Nonterminals (e.g., A, B)
- Σ is alphabet (e.g., a, b)
- R is RULES of form $A \rightarrow (N \cup \Sigma)^*$. (e.g., $S \rightarrow ASB$)
- S is the start Nonterminal.
- $L(G)$ is strings in Σ^* that can be generated.

Examples of CFGs

Find CFG's for the following

- $\{a^n : n \geq 0\}$
- $\{a^n : n \geq 10\}$
- $\{a^n b^m : n \leq m\}$.
- $\{a^n b^m : n < m\}$.

Do in class.

Examples of CFGs

Find CFG's for the following

- $\{a^n : n \geq 0\}$
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- $\{a^n b^m : n \leq m\}$.
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Do in class.

CFG that Deserves Proof

$\{w : \#_a(w) = \#_b(w)\}$.
Try to do it yourself first

CFG that Deserves Proof

$\{w : \#_a(w) = \#_b(w)\}$.

$S \to aSb$

$S \to bSa$

$S \to SS$

$S \to e$

Proof that it works

Do in class.

Constructive Induction

If you know or guess the form of the answer, you can sometimes derive the actual answer while doing mathematical induction to prove it.

Constructive Induction: Example

Example

For all $n \geq 1$

$$\sum_{i=1}^{n} 4i - 2 \; = \; ?$$

Guess that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} 4i - 2 \; = \; an^2 + bn + c \qquad \text{Why?}$$

Find constants a, b, and c such that this holds.

Do in class.

Constructive induction: Recurrence

Example

Let

$$a_n = \begin{cases} 2 & \text{if } n = 0 \\ 7 & \text{if } n = 1 \\ 12a_{n-1} + 3a_{n-2} & \text{if } n \geq 2 \end{cases}$$

What is a_n? Guess that for all integers $n \geq 0$,

$$a_n \;\leq\; AB^n \qquad \text{Why?}$$

Find constants A and B such that this holds:

Primarily find smallest B and secondarily smallest A.

$\boxed{\text{Do in class.}}$