How Many Ways can 1 Be Written as the Sum of Five Distinct Reciprocals?

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Abstract

I discuss a problem on reciprocls that was on a Maryland Math Competition.

1 Introduction

The following was problem number 2 (out of 5) on the *The University of Maryland High School Mathematics Competition*:

(a) The equality $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$ *and* $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42}=1$ *express 1 as the sum of three (respectively four) distinct positive integers. Find five positive integers* $a<b<c<d<e$ *such that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}=1.$$

(b) Prove that for any integer $m \geq 3$ *there exists* m *positive integers* $d_1 < d_2 < \cdots d_m$ *such that*

$$\frac{1}{d_1}+\cdots+\frac{1}{d_m}=1.$$

I graded this problem. This took longer than I thought it would since the (roughly) 200 students had 32 different correct solutions to part (a). (I got

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very good an adding reciprocals.) There were 4 different correct proofs and 1 very interesting but incorrect proof to part (b). In the appendix I present the solutions the students submitted to both parts (a) and (b) and discuss the connections between them.

Since there were 32 solutions submitted the question arose: (1) how many solutions were there? and (2) Could I do this by hand? The answers are (1) 74, and (2) yes but just barely. The question of determining how many ways 1 can be written as the sum of 6 reciprocals is (in my estimation) beyond reasonable hand calculation.

2 Student Solutions

We present the solutions to part (b) first. Within the proof we give other observations and student comments.

Theorem 2.1 *For all $n \geq 3$ there exists positive integers $d_1 < \cdots < d_n$ such that*

$$\frac{1}{d_1} + \cdots + \frac{1}{d_n} = 1.$$

Proof sketch:

We sketch four different correct solutions and an interesting failure. They are all by induction. We can assume the theorem is known for $n = 3$ and $n = 4$.

SOLUTION ONE: Use

$$\frac{1}{d} = \frac{1}{d+1} + \frac{1}{d(d+1)}.$$

This was the most common solution. This leads to (2,3,7,43,1806) for part a. Note that to do this the students would need to multiply 42 by 43.

SOLUTION TWO: Since we already have that the theorem is true for $n = 3$ and $n = 4$ we only need to prove that if it holds for n then it holds for $n + 2$. Use

$$\frac{1}{d} = \frac{1}{2d} + \frac{1}{3d} + \frac{1}{6d}.$$

This leads to (2,3,12,18,36) for part a. One of the students later told me that he knew SOLUTION ONE but used SOLUTION TWO to avoid having to multiply 42 by 43.

SOLUTION THREE: Inductively assume that the largest denominator d is even Use

$$\frac{1}{d} = \frac{3}{3d} = \frac{1}{3d} + \frac{2}{3d} = \frac{1}{3d} + \frac{1}{(3d/2)}.$$

Four students did the problem this way. This leads to (2,3,7,63,126) for part a.

SOLUTION FOUR: If

$$\frac{1}{d_1} + \cdots + \frac{1}{d_n} = 1$$

then

$$\frac{1}{2} + \frac{1}{2d_1} + \cdots + \frac{1}{2d_n} = 1$$

Only two student did the problem this way. It leads to (2,4,6,14,84) for part a which they both used.

ALMOST SOLUTION: Let A be a number that is the sum of n of its distinct divisors. (This is the problem with the solution: We need that, for all n, there is such an A.) Let $d_1, \ldots, d_n$ be the divisors of A that sum to A. Consider

$$\frac{d_1}{A} + \cdots + \frac{d_n}{A}.$$

Since, for all i, d_i divides A, this is a sum of distinct reciprocals. Since $\sum_{i=1}^{n} d_i = A$, the sum of these reciproacls is 1.

Could this solution work? The premise that it needs, that there is such an A that is the sum of n of its distinct divisors is true! How do we prove it. Alas, the only prove we know is by using the following theorem: For all n there exists a way to write 1 as a sum of n distint reciprocals.

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The students submited 32 correct solutions to part a. Several people had their part a and b not related to each other at all. I list all correct submitted solutions in lexicographic order along with how many students submitted each one.

1. (2,3,7,43,1806) - 91 (linked to SOLUTION ONE above)
2. (2,3,7,48,336) - 3

3. (2,3,7,56,168) - 1
4. (2,3,7,63,126) - 6 (linked to SOLUTION THREE above)
5. (2,3,7,70,105) - 1
6. (2,3,8,25,600) - 1
7. (2,3,8,30,120) - 1
8. (2,3,8,32,96) - 6
9. (2,3,8,36,72) - 5
10. (2,3,8,42,56) - 11
11. (2,3,9,21,126) - 2
12. (2,3,9,24,72) - 4
13. (2,3,9,27,54) - 3
14. (2,3,10,20,60) - 5
15. (2,3,11,22,33) - 1
16. (2,3,12,15,60) - 1
17. (2,3,12,16,48) - 1
18. (2,3,12,14,84) - 2 (linked to SOLUTION FOUR above)
19. (2,3,12,18,36) - 12 (linked to SOLUTION TWO above)
20. (2,4,5,25,100) - 3
21. (2,4,5,30,60) - 1
22. (2,4,6,14,84) - 3
23. (2,4,6,16,48) - 1
24. (2,4,6,18,36) - 2
25. (2,4,6,20,30) - 1

26. (2,4,7,12,42) - 4
27. (2,4,7,14,28) - 2 (Linked to the ALMOST SOLUTION above)
28. (2,4,8,12,24) - 6
29. (2,4,8,10,40) - 2
30. (2,5,6,10,30) - 1
31. (2,5,6,12,20) - 2
32. (3,4,5,6,20) - 3