The Coloring Square Theorem

William Gasarch

May 13, 2015

1 Introduction

For us a *square* is four lattice points in the first quadrant of the plane that form the vertices of a square.

Def 1.1 A *square* is a set of four points in the plane of the form $\{(x,y),(x,y+d),(x+d,y),(x+d,y+d)\}$ where $x,y,d \in \mathbb{N}$. If a coloring is understood then a *mono square* is a square where all of the points are the same color.

Consider the following theorem:

Theorem 1.2 *There exists M such that for all 2-colorings of $M \times M$ there exists a mono square.*

This theorem is not new. It follows from the Gallai-Witt theorem or the Hales-Jewitt theorem immediately, and it follows from Van Der Waerden's theorem with some work. There are also other proofs. We present an elementary proof that does not require any prior math; however, for those who know the proof of van der Waerde's theorem the ideas will look familiar. We believe the proof is original.

In this paper we give six proofs of this. They are in increasing order of how many prior theorems they need.

2 An Elementary Proof

For us an L is three lattice points in the first quadrant of the plane that form the vertices of an isosceles right triangle.

Def 2.1 An L is a set of three points in the plane of the form $\{(x,y),(x,y+d),(x+d,y)\}$ where $x,y,d \in \mathbb{N}$. If a coloring is understood then a *mono L* is an L where all of the points are the same color.

We prove the following lemma en route to proving the square theorem.

Lemma 2.2 *Let $a = 3 \times 512 = 1539$. For all 2-colorings of $a \times a$ there exists a mono L.*

Proof:

We assume all 2-colorings mentioned have no mono L. We will drive towards a contradiction.

Let COL be a finite coloring of a grid. A *mono almost-L* is one of the following colorings of an L:

$$\begin{array}{cccc} \cdots & R & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \cdots & R & \cdots & X \end{array}$$

$$\begin{array}{cccc} \cdots & B & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & B & \cdots & X \end{array}$$

The point where X is is called *the foot*. The *color of the L* is the color of the long side.

The following claim is easy to prove so we won't bother.

Claim 1: If COL is a 2-coloring of the 3×3 grid then there is a mono almost-L.

The following points are key not just to this proof but to many proofs in Ramsey Theorem:

Key 1: View the 1539×1539 grid of lattice points as a 513×513 grid of 3×3 grids.

Key 2: View a 2-coloring of the 1539×1539 grid as a 512-coloring of the a 513×513 grid of 3×3 grids.

Look at the left most column of this 513×513 grid. Two of the 3×3 grids have the same color. Those grids also have within them a *mono almost-L* in the same spot. We draw those two 3×3 grids AND the grid that is to the right the same distance:

$$
\begin{array}{cccccccc}
\cdots & R_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & R_2 & \cdots & B_1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & R_3 & \cdots & \cdots & \cdots & X & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots \\
\cdots & R_4 & \cdots & B_2 & \cdots & Y & \cdots & Z
\end{array}
$$

Look at the color of Z. If Z is red then R_1, R_4, Z form a monochromatic L. If Z is blue then B_1, B_2, Z form a monochromatic L. Hence there will be a monochromatic L. $\blacksquare$

What about 3 colors.

Lemma 2.3 *There exists a number a such that for all 3-colorings of $a \times a$ there exists a mono L.*

Proof:

We will use a 4×4 grid the same way that a 3×3 grid was used in the Lemma 2.3.

The following claim is easy to prove so we won't bother.

Claim 1: If COL is a 3-coloring of the 4×4 grid then there is a mono almost-L.

The following points are key not just to this proof but to many proofs in Ramsey Theorem:

Let b be a number to be picked later. We will make sure that 4 divides b; however, b will be quite large so this will not be an issue.

Key 1: View the $b \times b$ grid of lattice points as a $b/4 \times b/4$ grid of 4×4 grids.

Key 2: View a 3-coloring of the $b \times a$ grid as a 3^{16}-coloring of the a $b/4 \times b/4$ grid of 4×4 grids.

Look at the left most column of this $a/4 \times a/4$ grid. If we take $a/4 \geq 3^{16}+1$ then two of the 4×4 grids have the same color. Those grids also have within them a *mono almost-L* in the same spot. We draw those two 4×4 grids AND the grid that is to the right the same distance:

$$
\begin{array}{cccccccc}
\cdots & R_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & R_2 & \cdots & B_1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & R_3 & \cdots & \cdots & \cdots & X & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots \\
\cdots & R_4 & \cdots & B_2 & \cdots & Y & \cdots & Z
\end{array}
$$

Look at the color of Z. If Z is red then R_1, R_4, Z form a monochromatic L. If Z is blue then B_1, B_2, Z form a monochromatic L. Hence Z is green.

Now what?

We have shown the following awkward-to-state claim

Claim 2: For any 3-coloring of the $4(3^{16}+1) \times 4(3^{16}+1)$ grid there exists two mono almost-L of different colors and the same foot. Note that if the coloring has no mono L then the foot is a different color than either L. Hence we must have a configuration that looks like the following:

$$
\begin{array}{cccccccc}
\cdots & R & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & R & \cdots & B & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & R & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots \\
\cdots & R & \cdots & B & \cdots & \cdots & \cdots & G
\end{array}
$$

Let $b = 4(3^{16}+1) \times 4(3^{16}+1)$. Let a be a number to be picked later. We will make sure that b divides a; however, a will be quite large so this will not be an issue.

Key 1: View the $a \times a$ grid of lattice points as a $a/b \times a/b$ grid of $b \times b$ grids.

Key 2: View a 3-coloring of the $a \times a$ grid as a 3^{b^2}-coloring of the a $a/b \times a/b$ grid of $b \times b$ grids.

Look at the left most column of this $a/b \times a/b$ grid. If we take $a/b \geq 3^{b^2}+1$ then two of the $b \times b$ grids have the same color. Those grids also have within them two *mono almost-L*'s of different colors and the same foot in the same spot.

Hence we have the following (this picture is stylized- that is, if we have two R's adjacent it does not mean they really are adjacent, but two pairs that are adjacent have the same distance between them.)

$$
\begin{array}{cccccccccccc}
\cdots & R_1 & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & B_1 & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & B & \cdots & G & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & R & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & B & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R_2 & B_2 & \cdots & G & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & X
\end{array}
$$

If X is R then R_1, R_2, X form a mono L. If X is B then B_1, B_2, X form a mono L. If X is G then the three G's form a mono L. Hence there we have a mono L.

How big is a? We choose a to be $4(3^{b^2}+1)+1$ where $b = 4(3^{16}+1) \times 4(3^{16}+1)$. To get a sense of how large this is lets ignore some terms.

b is approx $16 \times 3^{32} \sim 3^{34}$.

a is approx $3^{b^2} = 3^{3^{34}}$. ∎

One can do a similar proof to obtain the following

Lemma 2.4 *For all c there is a number N_c such that for all c-colorings of the $N_c \times N_c$ grid there exists a mono L.*

We use Lemma 2.4 to prove our theorem.

Theorem 2.5 *There exists a number M such that for all 2-colorings of $M \times M$ there is a mono square.*

Proof: We will use a $N_2 \times N_2$ grid the same way that a 3×3 grid in Lemma X and a 4×4 grid in Lemma Y.

The following claim is true by the definition of N_2.

Claim 1: If COL is a 2-coloring of the $N_2 \times N_4$ grid then there is a mono L.

The following points are key not just to this proof but to many proofs in Ramsey Theorem:

Let M be a number to be picked later. We will make sure that N_2 divides M; however, M will be quite large so this will not be an issue.

Key 1: View the $a \times a$ grid of lattice points as a $M/N_2 \times M/N_2$ grid of $N_2 \times N_2$ grids.

Key 2: View a 2-coloring of the $a \times a$ grid as a $2^{N_2^2}$-coloring of the a $M/N_2 \times M/N_2$ grid of $N_2 \times N_2$ grids.

Take $M/N_2 = N_{2^{N_2^2}}$. Note that we have a $2^{N_2^2}$-coloring of the $N_{2^{N_2^2}} \times N_{2^{N_2^2}}$ grid. By the definition of N_c there is a mono L of $N_2 \times N_2$ grids. Each of these grids has a mono L in the same place and of the same color. Hence we have the following:

$$
\begin{array}{cccccc}
\cdots & R & \cdots & \cdots & W & X \\
\cdots & R & R & \vdots & Y & Z \\
\vdots & \vdots & \vdots & \vdots & \cdots & \\
\cdots & R & \cdots & \cdots & R & \\
\cdots & R & R & \cdots & R & R
\end{array}
$$

The reader can easily check that if any of W, X, Y, Z is R then there is a R mono square, and if they are all B then W, X, Y, Z form a B mono square.

How big does M have to be. We'll just say *very big* and leave it at that.

$\blacksquare$

3 What about if we 3-color?

Theorem 3.1 *There exists a number M such that for all 3-colorings of $M \times M$ there is a mono square.*

Proof:

We begin in a manner similar to the last theorem and come to the following point:

$$
\begin{array}{cccccc}
\cdots & R & \cdots & \cdots & W & X \\
\cdots & R & R & \vdots & Y & Z \\
\vdots & \vdots & \vdots & \vdots & \cdots & \\
\cdots & R & \cdots & \cdots & R & \\
\cdots & R & R & \cdots & R & R
\end{array}
$$

The reader can easily check that if any of W, X, Y, Z is R then there is a R mono square. So we now know that W is B or G, X is B or G, Y is B or G, Z is B or G.

We then use the grid that forced this to happen itself as a tile and get an L-shape of those. So we get

$$
\begin{array}{ccccccccccc}
\cdots & R & \cdots & \cdots & W & X & & & & W' & X' \\
\cdots & R & R & \vdots & Y & Z & & & & Y' & Z' \\
\vdots & \vdots & \vdots & \vdots & \cdots & & & & & & \\
\cdots & R & \cdots & \cdots & R & & & & & & \\
\cdots & R & R & \cdots & R & R & & & & & \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots\cdots\vdots & \vdots & \vdots & \vdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots\cdots\vdots & \vdots & \vdots & \vdots & \cdots & \\
\cdots & R & \cdots & \cdots & W & X\cdots\cdots\cdots & R & \cdots & \cdots & W & X \\
\cdots & R & R & \vdots & Y & Z\cdots\cdots\cdots & R & R & \vdots & Y & Z \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots\cdots\vdots & \vdots & \vdots & \vdots & \cdots & \\
\cdots & R & \cdots & \cdots & R & \cdots\cdots\cdots & R & \cdots & \cdots & R & \\
\cdots & R & R & \cdots & R & R\cdots\cdots\cdots & R & R & \cdots & R & R
\end{array}
$$

Case 1: One of W, X, Y, Z is RED. Then we have a RED square.

Case 2: We can assume that NONE OF W, X, Y, Z are RED. Assume X is BLUE. If X' is also BLUE then we have a BLUE square.

Case 3: None of W, X, Y, Z is RED and X is BLUE and X' is GREEN. Let the above be your tile. Take M large enough to get a mono L-shape of those tiles. Either there will be a mono square OR there will be a point that can't be RED or BLUE or GREEN. Hence there will be a mono square.