What is a predicate?

A "predicate" is a statement involving variables over a specified "domain" (set). Domain is **understood ahead of time**

Example

Domain (set)	Predicate
Integers ($\mathbb{Z}$)	$S(x)$: x is a perfect square
Reals ($\mathbb{R}$)	$G(x, y)$: $x > y$
Computers	$A(c)$: c is under attack
Computers; People	$B(c, p)$: c is under attack by p

Quantification

- Existential quantifier: $\exists x$ (exists x)
- Universal quantifier: $\forall x$ (for all x)

Domain D

- $(\exists x)[P(x)]$: Exists x in D such that $P(x)$ is true.
- $(\forall x)[P(x)]$: For all x in D, $P(x)$ is true.

If Domain not understood OR if want to use $D' \subseteq D$:

- $(\exists x \in D')[P(x)]$
- $(\forall x \in D')[P(x)]$

Examples From Mathematics

Domain is the natural numbers. Want to express everything just using $+, -, \times, \div, =, \leq, \geq$
Want predicates for

1. x is a odd.
2. x is a composite.
3. x is a prime.
4. x is the sum of three odd numbers.

Want to express **Vinogradov's Theorem:**
For every sufficiently large odd number is the sum of three primes.

Do All of this in class

Examples From Mathematics

Domain is the natural numbers. Want to express everything just using $+, -, \times, \div, =, \leq, \geq$

Want predicates for

1. x is a square.
2. When you divide x by 4 you get a remainder of 1.

Want to express
Theorem: Every prime that when you divide by 4 you get a remainder of 1 can be written as the sum of two squares.

Establishing Truth and Falsity

- To show $\exists$ statement is true:
  Find an example in the domain where it is true.
- To show $\exists$ statement is false:
  Show false for every member of the domain.
- To show $\forall$ statement is true:
  Show true for every member of the domain.
- To show $\forall$ statement is false:
  Find an example in the domain where it is false.

There are other methods!!!

Domain Matters

Is the following true:

$$(\forall x)(\exists y)[y < x]$$

If Domain is N? (Naturals)
If Domain is Z? (Integers)
If Domain is Q? (Rationals)
If Domain is $Q^{>0}$? (Rationals that are > 0)
If Domain is $Q^{\geq 0}$? (Rationals that are ≥ 0)
If Domain is R? (Reals)
If Domain is C? (Complex)

Negation of $\exists$ Statements: Cats

Domain is cats of students in CMSC 250H

It is not the case that some cat died

Can there exist a cat that died? **No**

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Hence **All cats are alive**

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Predicate $L(x)$: x is alive.

$$\neg(\exists x)[L(x)] \quad \equiv \quad (\forall x[\neg L(x)]$$

Negation of $\exists$-Statements: Math

Domain is the integers.

It is not the case that there exists A Pollard Number

Let $P(x)$ be that x is a Pollard Number

$$\neg(\exists x)[P(x)]$$

If there cannot exists a number x that is Pollard then

Negation of $\exists$-Statements: Math

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For all x, x is NOT Pollard.

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If there cannot exists a number x that is Pollard then
For all x, x is NOT Pollard.

$$(\forall x)[\neg P(x)].$$

Negation of $\exists$-Statements: Upshot

Any Domain D. Any Prediate P.

$$\neg(\exists x)[P(x)] \equiv (\forall x)[\neg P(x)]$$

Negation of $\forall$ Statements: Cats

Domain is cats of students in CMSC 250H

It is not the case that all cats are furry

Can there exist a cat that is not furry? **Yes**

Negation of $\forall$ Statements: Cats

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Can there exist a cat that is not furry? **Yes**

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It is not the case that all cats are furry

Can there exist a cat that is not furry? **Yes**

Hence **There is some cat that is not furry**

Predicate $F(x)$: x is furry.

$$\neg(\forall x)[F(x)] \equiv (\exists x[\neg F(x)]$$

Negation of $\forall$-Statements: Math

Domain is the integers.

It is not the case that all numbers are inteesting

Let $I(x)$ be that x is a Interesting.

$$\neg(\forall x)[I(x)]$$

NOT all numbers are interesting.

Negation of $\forall$-Statements: Math

Domain is the integers.

It is not the case that all numbers are inteesting

Let $I(x)$ be that x is a Interesting.

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NOT all numbers are interesting.

There must exist a number that is not interesting.

Negation of $\forall$-Statements: Math

Domain is the integers.

It is not the case that all numbers are inteesting

Let $I(x)$ be that x is a Interesting.

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NOT all numbers are interesting.

There must exist a number that is not interesting.

$$(\exists x)[\neg I(x)].$$

Note: We return to interesting numbers later.

Negation of $\forall$-Statements: Upshot

Any Domain D. Any Prediate P.

$$\neg(\forall x)[P(x)] \equiv (\exists x)[\neg P(x)]$$

Aside: $(\forall x)[I(x)]$??

0: $(\forall x)[x + 0 = x]$.

1: $(\forall x)[x \times 1 = x]$.

2: the only even prime.

3: the number of dimensions we live in.

4: Can color all planar maps with 4 colors but not 3.

5: The number of platonic solids.

6: The smallest perfect number.

7: Least n so can't contruct a reg polygon on n sides.

8: The largest cube in the Fibonacci sequence.
9: Max number of 3-powers are needed to sum to any integer.

Aside: $(\forall x)[I(x)]$??

10: The base of our number system.

11: The max mult persistence of a number

12: The smallest abundant number.

13: The number of Archimedian Solids.

14: There is NO n with exactly 14 numbers rel prime to it.

15: Least comp number with only one group of its order.

16: $= 2^4 = 4^2$. Only number that is x^y and y^x.

17: The number of wallpaper groups.

18: Only number that is twice the sum of its digits.

19: Max number of 4-powers needed to sum to any integer.

All Nat Numbers are Interesting

Assume $(\exists n)[\neg I(n)]$.
Let n be the least such n.

All Nat Numbers are Interesting

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WOW- n is the least number that is not interesting!

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Thats fascinating! so n is interesting!

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Contradiction

Vacuous cases for universally quantified statements

- All even prime numbers that are greater than 10 are the sum of two squares.
- All students in this class who are more than ten feet tall have green hair.
- All people associated with CMSC 250H who are not awesome have purple hair.
- All people associated with CMSC 250H who are not awesome have brown hair.

Are these statements True or False?
How do we show it?

Does Order Matter for $(\exists x)(\exists y)$?

$(\exists x)(\exists y)[x + y = 0]]$
$(\exists y)(\exists x)[x + y = 0]]$
Equivalent? Vote!

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Let σ be a 1-1 onto function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

$$(\exists x_1)(\exists x_2) \cdots (\exists x_n)[P(x_1, \ldots, x_n] \equiv$$

$$(\exists x_{\sigma(1)})(\exists x_{\sigma(2)}) \cdots (\exists x_{\sigma(n)})[P(x_1, \ldots, x_n]$$

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Equivalent? Vote!

NOT equivalent. We fine a domain where one is TRUE and one is FALSE.

$(\forall x)(\exists y)[x < y]]$ TRUE over Naturals.
$(\exists y)(\forall x)[x < y]]$ FALSE over Naturals.

Logical Rules

Universal Instantiation	$\dfrac{(\forall x)[P(x)]}{\therefore P(c)}$
Universal Generalization	$\dfrac{P(c) \text{ for arbitrary element } c}{\therefore (\forall x)[P(x)]}$
Existential Instantiation	$\dfrac{(\exists x)[P(x)]}{\therefore P(c) \text{ for some element } c}$
Existential Generalization	$\dfrac{P(c) \text{ for some element } c}{\therefore (\exists x)[P(x)]}$