Homework 4, MORALLY Due Feb 25

WARNING: THIS HW IS TWO PAGES LONG!!!!!!!!!!!!!!!!!

1. (20 points) Simplify the following formula so that its of the form QUANTIFIER QUANTIFIER then stuff. In other words, there is no negation on the outside or between the quantifiers.

$$\neg(\forall x)(\exists y)[R(x,y) \wedge \neg S(x,y)].$$

SOLUTION TO PROBLEM 1

$$(\exists x)(\forall y)[\neg(R(x,y) \vee S(x,y)]$$

2. (30 points) The domain is N which includes 0.

   (a) (5 points) Write an expression $SQ(x)$ which will mean that x is a square.

   (b) (5 points) Write an expression $SUMSQ2(x)$ which will mean that x is the sum of two squares.

   (c) (5 points) For all n show how you can write an expression $SUMSQn(x)$ which will mean that x is the sum of n squares. Use SQ.

   (d) (5 points) Write a sentence that means that every natural is the sum of 1, 2, or 3 squares. Use the predicates you have defined above.

   (e) (0 points but you will need this for the next part). Write a program that will, for all $0 \le x \le 1000$ determine the smallest number of squares such that x is the sum of that many squares. (For this part do not hand anything in.)

   (f) (15 points) Based on the data you produces make TWO conjectures along the lines of:

   - Every number is the sum of at most BLAH squares.
   - The infinite set X is such that every number in X can be written as the sum of BLAH squares but NOT BLAH-1 squares. (NOTE- X should be a nice set– its okay if some elements NOT in X also need BLAH squares.)

SOLUTION TO PROBLEM 2

a) $SQ(x)$ is

$$(\exists y)[x = y^2].$$

b) $SUMSQ2(x)$ is

$$(\exists x_1)(\exists x_2)[SQ(x_1) \wedge SQ(x_2) \wedge x = x_1 + x_2].$$

c) $SUMSQn(x)$ is

$$(\exists x_1, \ldots, x_n)[SQ(x_1) \wedge \cdots \wedge SQ(x_n) \wedge x = x_1 + \cdots + x_n].$$

e)

Every number is the sum of FOUR squares

Every number of the form $4^m(8n+7)$ can be written as the sum of FOUR squares but NOT THREE squares.

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3. (20 points) For the following sentences find both (a) an infinite domain where it is true, and (b) an infinite domain where it is false. All domains should be subsets of R.

   $(\forall x)(\exists y)[x = y^2]$ but DO NOT use R or any closed or open or clopen interval.

   SOLUTION TO PROBLEM 3

   Let

   $D_0 = Z$

   $D_1 = D_0 \cup \{\sqrt{x} \mid x \in D_0\}$

   For all $i \geq 2$

   $D_i = D_0 \cup \{\sqrt{x} \mid x \in D_0 \cup \cdots \cup D_{i-1}\}$

   Now let

$$D = D_0 \cup D_1 \cup \cdots$$

   Every number in D_i has a square root in D_{i+1}, hence every element of D has a square root in D.

4. (30 points) (Recall that Q is the rationals.)

   (a) Prove that $\sqrt{5} \notin Q$ using the mod method. (Hint: First prove a lemma about mods.)

   (b) Prove that $\sqrt{5} \notin Q$ using unique factorization.

   SOLUTION TO PROBLEM 4

   a)

   Lemma: If $x^2 \equiv 0 \pmod 5$ then $x \equiv 0 \pmod 5$.

   Proof: First take the contrapositive

   If $x \not\equiv 0 \pmod 5$ then $x^2 \not\equiv 0 \pmod 5$.

   We do this by cases. All $\equiv$ are mod 5.

   If $x \equiv 1$ then $x^2 \equiv 1^2 \equiv 1 \not\equiv 0$

   If $x \equiv 2$ then $x^2 \equiv 2^2 \equiv 4 \not\equiv 0$

   If $x \equiv 3$ then $x^2 \equiv 3^2 \equiv 9 \equiv 4 \not\equiv 0$

If $x \equiv 4$ then $x^2 \equiv 4^2 \equiv 16 \equiv 1 \not\equiv 0$

End of Proof

Theorem: $\sqrt{5} \notin Q$.

Proof: Assume, by way of contradiction, that $\sqrt{5} \in Q$. Hence there exists a, b IN LOWEST TERMS such that

$\sqrt{5} = \frac{a}{b}$

$5 = \frac{a^2}{b^2}$

$5b^2 = a^2$

So $a^2 \equiv 0 \pmod 5$. By Lemma $a \equiv 0 \pmod 5$. Let $a = 5c$

$5b^2 = a^2$ is now

$5b^2 = (5c)^2 = 25c^2$

$b^2 = 5c^2$.

So $b^2 \equiv 0 \pmod 5$. By Lemma $b \equiv 0 \pmod 5$.

We now have that a and b both have a factor of 5. Hence a, b are NOT in lowest terms. This CONTRADICTS that a, b are not in lowest terms.

2)

Theorem: $\sqrt{5} \notin Q$.

Proof: Assume, by way of contradiction, that $\sqrt{5} \in Q$. Hence there exists a, b such that:

$\sqrt{5} = \frac{a}{b}$

$5 = \frac{a^2}{b^2}$

$5b^2 = a^2$

We FACTOR a, b:

$a = p_1^{a_1} \cdots p_L^{a_L}$

so $a^2 = p_1^{2a_1} \cdots p_L^{2a_L}$

$b = p_1^{b_1} \cdots p_L^{b_L}$

so $b^2 = p_1^{2b_1} \cdots p_L^{2b_L}$

(NOTE- some of the a_i's and b_i's could be 0.)

$5b^2 = a^2$

so

$$5p_1^{2b_1} \cdots p_L^{2b_L} = p_1^{2a_1} \cdots p_L^{2a_L}$$

Be reordering let $p_1 = 5$.

The number of 5's on the LHS is $2b_1 + 1$.

The number of 5's on the RHS is $2a_1$.

Hence

$2b_1 + 1 = 2a_1$

$1 = 0$

CONTRADICTION.

END OF PROOF