Homework 5, MORALLY Due March 25

WARNING: THIS HW IS TWO PAGES LONG!!!!!!!!!!!!!!!!!!

1. (20 points)

   (a) (0 points) What is $\int_1^n x^2 dx$?

   (b) (2 points) Use the answer to part 1 to conjecture the FORM of the formula (with some of the coefficients not know) for

$$\sum_{i=1}^{n} i^2$$

   (To make your life easier you can also conjecture what the first coefficient is based on the interval.)

   (c) (9 points) (Use part b) By plugging in $n = 0$, $n = 1$, and perhaps more find a very good guess for the formula for $\sum_{i=1}^{n} i^2$. Show your work of course. (It should be the real formula.)

   (d) (9 points) (Use part b) Derive the formula by constructive induction.

SOLUTION TO PROBLEM 1

(a) What is $\int_1^n x^2 dx$?

$$\int_1^n x^2 dx = \frac{n^3}{3} - \frac{1}{3}$$

(b) Use the answer to part 1 to conjecture a formula (with some of the coefficients not know) for

$$\sum_{i=1}^{n} i^2$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + Bn^2 + Cn + D$$

(c) By plugging in $n = 0$, $n = 1$, and perhaps more find a very good guess for the formula for $\sum_{i=1}^{n} i^2$.

$n = 0$: $0 = D$. Great! we already know D. One less coefficient to drag around
$n = 1$: $1 = \frac{1}{3} + B + C$. We rewrite as $B + C = \frac{2}{3}$ and then as

$$3B + 3C = 2.$$

$n = 2$: $1^2 + 2^2 = \frac{8}{3} + 4B + 2C$
so $5 = \frac{8}{3} + 4B + 2C$. We rewrite as $\frac{7}{3} = 4B + 2C$ and then as

$$12B + 6C = 7.$$

We rewrite the two equations:

$$\begin{aligned} 3B + 3C &= 2 \\ 12B + 6C &= 7 \end{aligned}$$

Multiply the first equation by 4

$$\begin{aligned} 12B + 12C &= 8 \\ 12B + 6C &= 7 \end{aligned}$$

Subtract the second from the first equation to get $6C = 1$ so $C = \frac{1}{6}$.
Then use $B = \frac{2}{3} - C = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$.
Hence

$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$= \frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6}$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$\frac{n(2n^2 + 3n + 1)}{6} = \frac{n(2n+1)(n+1)}{6}$$

(d) We want to prove

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + Bn^2 + Cn + D$$

But we do not know what B, C, D are. We DO the prove and rather than say at a certain stage TRUE we instead state a condition on B, C, D. At the end we hope to find a B, C, D that satisfy all of the constraints.

Base Case: $n = 0$. Get $0 = 0 + 0 + D$. So $D = 0$ is a constraint. We will just set $D = 0$ in the next step.

IH: $\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + Bn^2 + Cn$

IS:

Start with $\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + Bn^2 + Cn$

ADD $(n+1)^2$ to both sides to get

Start with $\sum_{i=1}^{n+1} i^2 = \frac{1}{3}n^3 + Bn^2 + Cn + (n+1)^2$

We WANT

$$\frac{1}{3}n^3 + Bn^2 + Cn + (n+1)^2 = \frac{1}{3}(n+1)^3 + B(n+1)^2 + C(n+1)$$

Expand both sides and set the coefficients of n^3 equal (this will just be $\frac{1}{3} = \frac{1}{3}$) the coefficients of n^2 equal, the coefficient of n equal, the constant term equal.

This gives 3 linear equations in 2 variables, however, they work out.

2. (20 points) Recall our usual *induction scheme*:

From

- $P(0)$
- $(\forall n \geq 0, n \in \mathsf{N})[P(n) \rightarrow P(n+1)]$

we get $(\forall n \in \mathsf{N})[P(n)]$.

This problem is about how to modify this scheme.

(a) (7 points) Give a scheme that will show

$$(\forall n \equiv 0 \pmod 4, n \in \mathsf{N})[P(n)].$$

(b) (7 points) Give a scheme that will show

$$(\forall n \equiv 0,1 \pmod 4, n \in \mathsf{N})[P(n)].$$

(c) (6 points) Give a scheme that will show

$$(\forall n \in \mathsf{Z})[P(n)].$$

SOLUTION TO PROBLEM 2

a)

- $P(0)$
- $(\forall n \geq 0)[P(n) \rightarrow P(n+4)]$

b)

- $P(0)$
- $P(1)$
- $(\forall n \geq 0)[P(n) \rightarrow P(n+4)]$

c)

- $P(0)$
- $(\forall n \geq 0)[P(n) \rightarrow P(n+1)]$
- $(\forall n \leq 0)[P(n) \rightarrow P(n-1)]$

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3. (20 points) Assume that there are constants A, B, C, D such that

$$(\forall n \geq 0, n \in \mathsf{N}) \left[\sum_{i=1}^{n} i \times 2^i = An2^n + B2^n + Cn + D \right].$$

(a) (10 points) Find A, B, C, D by plugging in $n = 0, 1, 2, 3$ (or less if you don't need all of those) into the equation.

(b) (10 points) Find A, B, C, D by constructive induction.

(c) (0 points- don't hand in) What are the PROS and CONS of each technique?

(d) (0 points- don't hand in) How could you have guessed the form of the summation above? One way is with integrals. Can you think of another way?

a) Plug in $n = 0$ to get

$$0 = A \times 0 \times 2^0 + B \times 2^0 + C \times 0 + D.$$

So we know that $0 = B + D$.

Plug in $n = 1$ to get

$$2 = A \times 1 \times 2^1 + B \times 2^1 + C \times 1 + D.$$

$$2 = 2A + 2B + C + D$$

Since we know that $B + D = 0$ we can simplify this to

$$2 = 2A + B + C$$

Plug in $n = 2$ to get

$$1 \times 2^1 + 2 \times 2^2 = A \times 2 \times 2^2 + B \times 2^2 + C \times 2 + D$$

$$10 = 8A + 4B + 2C + D$$

Since we know $B + D = 0$ we can simplify this to

$$10 = 8A + 3B + 2C$$

Plug in $n = 3$ to get

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 = A \times 3 \times 2^3 + B \times 2^3 + C \times 3 + D$$

$$34 = 24A + 8B + 3C + D$$

Since we know $B + D = 0$ we can simplify this to

$$34 = 24A + 7B + 3C$$

We rewrite the $n = 1, 2, 3$ equations which do not have D.

$$2 = 2A + B + C$$

$$10 = 8A + 3B + 2C$$

$$34 = 24A + 7B + 3C$$

From the first two equations we get

$$6 = 4A + B$$

If you add the first two equations and subtract from the third you get

$$22 = 14A + 9B$$

We now want to eliminate the B so multiply the first equation by 9 and subtract the second

$$32 = 22A$$

2) By Constructive Induction.

Base Case: $n = 0$ yields $B + D = 0$

IH:

$$\sum_{i=1}^{n} i \times 2^i = An2^n + B2^n + Cn + D.$$

IS: Take the IH and add $(n+1)2^{n+1}$ to both sides.

$$\sum_{i=1}^{n+1} i \times 2^i = An2^n + B2^n + Cn + D + (n+1)2^{n+1}.$$

So we WANT

$$An2^n + B2^n + Cn + D + (n+1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C(n+1) + D$$

$$An2^n + B2^n + Cn + (n+1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C(n+1)$$

$$An2^n + B2^n + (n+1)2^{n+1} = An2^{n+1} + A2^{n+1} + B2^{n+1} + C$$

$$(n+1)2^{n+1} = An2^n + A2^{n+1} + B2^n + C$$

$$(2n)2^n + 2(2^n)) = An2^n + (2A + B)2^n + C$$

$A = 2$, $B = -2$, $C = 0$.

Since $B + D = 0$, $D = 2$ So we get

$$\sum_{i=1}^{n} i \times 2^i = 2n2^n + -2 \times 2^n + 2.$$

4. (20 points) Let $T(n)$ be defined by

$$T(1) = 10$$

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 17n$$

By constructive induction find value $c \in \mathsf{N}$ such that $(\forall n)[T(n) \le cn$. Try to make c as small as possible (and its in N so this is possible).

SOLUTION TO PROBLEM 4

Base Case: $n = 1$: $T(1) = 10$ so need $10 \le c \times 1 = c$.

IH: For all $n' < n$, $T(n') \le cn'$.

IS:

$$T(n) = T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lfloor \frac{n}{3} \right\rfloor) + 17n$$

$$\le c\left\lfloor \frac{n}{2} \right\rfloor + c\left\lfloor \frac{n}{3} \right\rfloor + 17n$$

$$\le \frac{cn}{2} + \frac{cn}{3} + 17n = \frac{5cn}{6} + 17n$$

WANT

$$\frac{5cn}{6} + 17n \le cn$$

$$17n \le \frac{cn}{6}$$

$$17 \le \frac{c}{6}$$

$$c \ge 17 \times 6 = 102$$

So need $c \ge 102$ and from the base case $c \ge 10$.

So take $c = 102$.

5. (20 points) In the country of Fredonia they only use 10-cent coins and 11-cent coins. Note that the people cannot have 9 cents on them, they can have 10, they can have 11, but they can't have 12.

   (a) (0 points and don't hand anything in) Write a program that will, for $n = 1$ to 1000, determine which numbers of cents good people of Fredonia can have.

   (b) (5 points) Make a conjecture of the form:

      - $n_0 - 1$ CANNOT be written in the form $10x + 11y$ with $x, y \in \mathsf{N}$.
      - $(\forall n \geq n_0)(\exists x, y \in \mathsf{N})[n = 10x + 11y]$.

      (So you need to find n_0.)

   (c) (15 points) Prove your conjecture by induction.

SOLUTION TO PROBLEM FIVE

We skip the programming and state what you should have discovered: $n_0 = 90$

PART ONE: 89 CANNOT be written as $10x + 11y$.

Case 1: If $x \geq 9$ then $10x + 11y \geq 90 > 89$.

Case 2: If $y \geq 9$ then $10x + 11y \geq 99 > 89$.

Case 3: $x \leq 8$ and $y \leq 8$. If

$$10x + 11y = 89$$

then taking the equation mod 10 you get

$$y \equiv 9 \pmod{10}$$

Since $0 \leq y \leq 8$ this cannot occur.

(OH-looks like Case 1 was not needed. Oh well.)

$(\forall n \geq 90)(\exists x, y \in \mathsf{N})[n = 10x + 11y]$.

BEFORE I begin lets to through our thought process.

If $n-1 = 10x' + 11y'$ and $x' \geq 1$ then can swap out a 10-cent coin and swap in an 11 cent coin.

But what about swapping out 11's for 10's? If we swap out 9 11-cent coins and swap in 10 10-cent coins then we are plus 1.

Base Case:

$90 = 10 \times 9 + 11 \times 0$.

IH: $n \geq 90$. For all $n' < n$ there exists x, ysuch that $n = 10x + 11y$.

IS: We prove for $n+1$

Case 1: $x \geq 1$. Then swap out a 10-cent coin and swap in an 11-cent coin to get

$$n + 1 = 10(x-1) + 11(y+1)$$

Case 2: $y \geq 9$. Then swap out a 9 11-cent coin and swap in 10 10-cent coin to get

$$n + 1 = 10(x+10) + 11(y-9)$$

Case 3: $x \leq 0$ and $y \leq 8$. Then

$$n = 10x + 11y \leq 10 \times 0 + 11 \times 8 = 88$$

This is a contradiction since $n \geq 90$. Hence Case 3 cannot occur.