Homework 8, Morally due Tue Apr 16, 3:30PM

THIS HW IS TWO PAGES!!!!!!!!!!!

Throughout this HW:

- Let $f(m,s)$ be the muffin function (from the talk Bill gave on Muffins).
- To prove that, say $f(11,5) = \frac{13}{30}$ you would need to BOTH give a PROCEDURE that allocates 11 muffins to 5 people with smallest piece $\frac{13}{30}$ AND prove that there is no BETTER procedure.
- You CANNOT use the Floor-Ceiling Theorem, though you can use the same kind of reasoning in a particular case.

1. (50 points) Prove $f(9,5) = \frac{2}{5}$.

   SOLUTION TO PROBLEM ONE

   Procedure that shows $f(9,5) \ge \frac{2}{5}$:

   (a) Divide 9 muffins $\{\frac{2}{5}, \frac{3}{5}\}$.
   (b) Give 3 students $\{\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}\}$.
   (c) Give 2 students $\{\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\}$.

   We can't do better: Assume there is a protocol with smallest piece $> \frac{2}{5}$.

   If some muffin is cut into ≥ 3 pieces then some piece is $\le \frac{1}{3}$. Hence we can assume every piece is cut into ≤ 2 pieces.

   If some muffin is uncut then one could just cut that muffin $\frac{1}{2}$-$\frac{1}{2}$ give the recipient both halfs.

   Hence we have that every muffin is cut into 2 pieces. So there are 18 pieces.

   Since there are 5 students, some student gets $\ge \lceil \frac{18}{5} \rceil = 4$ pieces. that student has a piece of size $\le \frac{9}{5} \times \frac{1}{4} = \frac{9}{20}$. We want $\le \frac{2}{5} = \frac{8}{20}$. DARN.

   Since there are 5 students, some student gets $\le \lfloor \frac{18}{5} \rfloor = 3$ pieces. that student ha a piece of size $\ge \frac{9}{5} \times \frac{1}{3} = \frac{3}{5}$. Look at the muffin that piece came from. The other prat of that muffin is of size $\le 1 - \frac{3}{5} = \frac{2}{5}$.

GO TO THE NEXT PAGE

2. (50 points) Prove $f(7,6) = \frac{1}{3}$.

SOLUTION TO PROBLEM TWO

a) $f(7,6) \ge \frac{1}{3}$

(a) Divide 3 muffins $(\frac{1}{2}, \frac{1}{2})$.

(b) Divide 4 muffins $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

(c) Give 6 students $[\frac{1}{2}, \frac{1}{3}, \frac{1}{3}]$

Assume, by way of contradiction, that $f(7,6) > \frac{1}{3}$. So there is a protocol where every piece is $> \frac{1}{3}$.

Case 1: Some muffin is cut into ≥ 3 pieces. Then some piece is $\le \frac{1}{3}$.

Case 2: Some muffin is uncut. Again, one can cut each uncut muffin 1/2-1/2 and give the two halfs to the recipient.

Case 3: All muffins are cut into exactly two pieces. So there are 14 pieces.

ATTEMPT AT THE USUAL ARGUMENT:

Since there are 6 students, some student gets $\ge \lceil \frac{14}{6} \rceil = 3$ pieces. That student gets some piece of size $\le \frac{7}{6} \times \frac{1}{3} = \frac{7}{18}$. DARN

Since there are 6 students, some student gets $\le \lfloor \frac{14}{6} \rfloor = 2$ pieces. That student gets some piece of size $\ge \frac{7}{6} \times \frac{1}{2} = \frac{7}{12}$. That piece's buddy is of size $\le 1 - \frac{7}{12} = \frac{5}{12}$. DARN

So the usual method won't work. We need to use the HALF method.

We DO have 14 pieces.

Case 3a: Alice gets ≥ 4 shares. Then Alice has a piece $\le \frac{7}{6} \times \frac{1}{4} = \frac{7}{24} < \frac{1}{3}$.

Case 3b: Alice gets ≤ 1 shares, so Alice gets 1 share. Alice gets $\frac{7}{6}$ so 1 share is not enough!

Case 3c: Everyone gets 2 or 3 shares. Let s_2 (s_3) be the number of people who get 2 (3) shares.

$2s_2 + 3s_3 = 14$

$s_2 + s_3 = 6$

SO $s_2 = 4$ and $s_3 = 2$.

A student who gets 2 shares is called a 2-student (and 3-shares is called a 3-student). A share that goes to a 2-student is a 2-share. Note that there are 8 2-shares and 6 3-shares.

Claim: All 2-shares are $> \frac{1}{2}$.

Proof: Assume Alice has a 2-share $\le \frac{1}{2}$. Then the other 2-share Alice has is $\ge \frac{7}{6} - \frac{1}{2} = \frac{2}{3}$. That pieces's buddy is $\le 1 - \frac{2}{3} = \frac{1}{3}$.

End of Proof of Claim

All 2-shares are $> \frac{1}{2}$. There are 8 2-shares. Hence there are 8 pieces that are $> \frac{1}{2}$. This is impossible since 7 muffins were cut into two pieces so there are at most 7 piece $> \frac{1}{2}$.