Homework 11, Morally due Tue May 7, 3:30PM

THIS HW IS THREE PAGES!!!!!!!!!!

THROUGHOUT THIS HW: Unless I tell you what n is, assume n is large enough so that we can use the approximations and the Ind assumptions we made in class (If you were not in class then — too bad for you!)

THROUGHOUT THIS HW: Assume the grader does not already know the material so clearly explain what you are doing ohow you are setting up your program.

1. (0 points but if you don't show up to the final I will assume you got this problem wrong and you will get 0 points for this entire HW) WHEN IS THE FINAL? WHERE IS THE FINAL?

   SOLUTION TO PROBLEM ONE

   Saturday May 18, 4-6 in PHYSICS 1201

2. (40 points) We are going to put m balls into n boxes, uniformly at random. You can assume n is large so our approximations work.

   (a) (20 points) What is the probability (approx) that some box has at least 4 balls in it?

   (b) (20 points) Make a statement:
   If $m \ge XXX$ then the prob that some box has 4 balls in it is $\ge \frac{3}{4}$.
   FILL IN THE XXX. XXX will depend on n. Try to make XXX as small as possible. Please provide a number for the constant in-front of n.

   SOLUTION TO PROBLEM TWO

   1) Denote $X_{i,j,k,\ell} = 1$ if the balls i, j, k, ℓ all fall into the same bin. Thus we want to compute $\Pr[\exists (i,j,k,\ell)\ \ X_{i,j,k,\ell} = 1]$. We first compute $\Pr[\forall (i,j,k,\ell)\ \ X_{i,j,k,\ell} = 0]$ which is the probability of the complement of the event. Approximating this via independence this is approximately equal to $\left(1 - \frac{1}{n^3}\right)^{\binom{m}{4}}$. This is approximately $\left(1 - \frac{1}{n^3}\right)^{m^4/24}$. Using the approximation $1 - x \approx e^{-x}$ when x is large, we get this to be $e^{-\frac{m^4}{24n^3}}$. Therefore the probability that some box has at least 4 balls is $1 - e^{-\frac{m^4}{24n^3}}$.

   2) For this to be greater than 3/4, we want $1 - e^{-\frac{m^4}{24n^3}} \ge 3/4$. Rearranging and taking log's we get $m \ge (48 \ln 2)^{0.25} n^{3/4} \sim 2.4 n^{3/4}$.

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3. (30 points) Assume n is large enough for our approximations to work. Also assume that $n^{0.9}$ is an integer. We are going to put $n^{0.9}$ balls into n boxes, uniformly at random. (Yes you read that right.)

   (a) What is the prob that some box has ≥ 2 balls?

   (b) What is the prob that some box has ≥ 3 balls?

   (c) Find a CONSTANT XXX (So XXX is ind of n) such that the prob that some box has $\ge XXX$ balls is $\ge \frac{1}{2}$. Try to make XXX as large as possible. (You may use that $\binom{n}{k} \sim n^k$.)

   SHOW all of your work so that if the grader is not familiar with the problem he can still understand your answer.

   SOLUTION TO PROBLEM THREE

   The number of ways to put $n^{0.9}$ balls into n boxes is $n^{n^{0.9}}$.

   1) The number of ways to put $n^{0.9}$ balls into n boxes so that no 2 are in the same box is

$$n(n-1)\cdots(n-n^{0.9}+1)$$

   So the prob is

$$\frac{n(n-1)\cdots(n-n^{0.9}+1)}{n^{n^{0.9}}}$$

$$= \frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-n^{0.9}+1}{n}$$

$$= 1 \times \left(1 - \frac{1}{n}\right) \times \left(1 - \frac{2}{n}\right) \times \cdots \times \left(1 - \frac{n^{0.9}}{n}\right)$$

   (We omit the last term in the product as an approx, get rid of the pesky $+1$.)

$$\approx e^{-(1+2+\ldots+n^{0.9})/n}$$

   Recall that $1 + 2 + \cdots + n^{0.9} \sim \frac{n^{1.8}}{2}$. Hence we have

$$\approx e^{-(1+2+\ldots+n^{0.9})/n} \approx e^{-n^{1.8}/2n} = e^{-n^{0.8}/2}$$

Therefore, prob. that some box has at least 2 is $1 - e^{-n^{0.8}/2}$.

2) Let $X_{i,j,k} = 1$ iff balls i, j, k fall into the same bin. As before, we want to compute $\Pr[\exists(i, j, k)\ \ X_{i,j,k} = 1]$. We will compute $\Pr[\forall(i, j, k)\ \ X_{i,j,k} = 0]$. Approximating this by independence, we get this is approximately $\left(1 - \frac{1}{n^2}\right)^{n^{2.7}/6}$. Thus, the required probability is $1 - \left(1 - \frac{1}{n^2}\right)^{n^{2.7}/6} \approx 1 - e^{-n^{0.7}/6}$.

3) For the purposes of this question, we will use $\binom{n}{k} \sim n^k$. Note that a better approx known as Stirling's approximation is known, which we ignore for this question.

Using the approximations above, we can obtain the probability that some box has at least k balls is

$$\sim 1 - e^{-\binom{n^{0.9}}{k}/n^{k-1}} \sim 1 - e^{-n^{0.9k}/n^{k-1}} = 1 - e^{-n^{1-0.1k}}$$

We want to solve for the inequality,

$$1 - e^{-n^{1-0.1k}} \geq 1/2.$$

$$\frac{1}{2} \geq e^{-n^{1-0.1k}}$$

$$e^{n^{1-0.1k}} \geq 2$$

Take ln of both sides to get

$$n^{1-0.1k} \geq \ln(2)$$

That was so much fun, lets do it again!

$$(1 - 0.1k) \ln n \geq \ln(\ln 2))$$

$$1 - 0.1k \geq \frac{\ln(\ln 2)}{\ln n}$$

$$1 \geq 0.1k + \frac{\ln(\ln 2)}{\ln n}$$

$$10 \geq k + \frac{10\ln(\ln 2)}{\ln n}$$

$$k \leq 10 - \frac{10\ln(\ln 2)}{\ln n}$$

$10\ln(\ln(2)) \sim -3.66$ so we have

$$k \leq 10 + \frac{3.66}{\ln n}$$

We can take $k = 10$.

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4. (30 points) Let p_n be the EXACT answer to the n-person hatcheck problem. Recall that

$$p_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}.$$

Write a program that does the following

For $2 \le n \le 20$

(a) Compute p_n exactly. (To deal with the precision issues in your code, please keep the numerator and denominator of the computed fractions separately and perform only one division in the end.)

(b) Compete $\frac{1}{e} - p_n$ (it may be positive or negative).

(c) Put your results in a nice easy-to-read tables with first column n and the second column p_n and third column $\frac{1}{e} - p_n$.

(d) Make a conjecture about when the $\frac{1}{e} - p_n$ is positive and when it is negative.

(e) Make a conjecture about how quickly $|\frac{1}{e} - p_n|$ goes to zero. For example, maybe you think $|\frac{1}{e} - p_n| \sim \frac{\pi}{n}$. (HINT: I do not think that.)

Your answer should have the columns and the conjectures.

SOLUTION TO PROBLEM FOUR

Omitted.