Homework 12, Morally due Tue May 14, 3:30PM

THIS HW IS ONE PAGES!!!!!!!!!!

WHEN IS THE FINAL? Saturday May 18, 4-6

WHERE IS THE FINAL? PHYSICS 1201

1. (30 points — 10 each) Show that the following sets are uncountable

   (a) The set of functions from N to N that are strictly increasing. (That means that, for all $x, y \in \mathsf{N}$, if $x < y$ then $f(x) < f(y)$.)

   (b) The set of functions from N to PRIMES.

   (c) The set of functions from N to PRIMES that are strictly increasing.

   SOLUTION TO PROBLEM ONE

   1) Assume, BWOC, that $f_1, f_2, f_3, \ldots$ is the set of ALL increasing functions.

   We CONSTRUCT a function that is increasing but is not one of $f_1, f_2, \ldots$.

   The function is F. We will make sure that $(\forall i)[F(i) \neq f_i(i)]$ and hence $(\forall i)[F \neq f_i]$. The trick will be to make sure that F is increasing.

   $F(0) = f_0(0) + 1$

   If $i \geq 1$ then let

   $$F(i) = \max\{F(0), \ldots, F(i-1), f_i(i)\} + 1$$

   F is clearly increasing AND, for all i, $F(i) \neq f_i(i)$.

   2) Assume, BWOC, that $f_1, f_2, f_3, \ldots$ is the set of ALL functions from N to the PRIMES

   We CONSTRUCT a function that is from N to PRIMES but is not one of $f_1, f_2, \ldots$. The trick will be to make sure that F only takes on values in the primes.

   $$F(i) = \text{ the next prime after } f_i(i)$$

   F clearly only takes on prime values AND, for all i, $F(i) \neq f_i(i)$.

3) Assume, BWOC, that $f_1, f_2, f_3, \ldots$ is the set of ALL increasing functions from N to the PRIMES.

We CONSTRUCT a function that is increasing and goes from N to PRIMES but is not one of $f_1, f_2, \ldots$.

The trick will be to make sure that F only takes on values in the primes AND is increasing.

$F(0) =$the next prime after $f_0(0)$.

If $i \geq 1$ then

$$F(i) = \text{ the next prime after } F(0), F(1), \ldots, F(i-1), f_i(i)$$

F clearly only takes on prime values, is increasing, AND, for all i, $F(i) \neq f_i(i)$.

2. (40 points — 20 points each) Let $(A, \leq_1)$ and $(B, \leq_2)$ be ordered sets. An *order preserving bijection f from A to B* is a bijection from A to B such that, for all $x, y \in A$.

$$x \leq_1 y \rightarrow f(x) \leq_2 f(y).$$

   (a) Show that there is NO order preserving bijections from N to Z.

   (b) Show that there is NO order preserving bijections from N to $\mathsf{Q}^{\geq 0}$. (Thats the rationals ≥ 0.)

SOLUTION TO PROBLEM TWO

1) Assume, BWOC, that there is an order preserving bijection f from N to Z. Let $f(0) = a$. Note that $a \in \mathsf{Z}$. Since f is a bijection from N to Z there exists $b \in \mathsf{N}$ such that $f(b) = a - 1$. Note that

$0 < b$ and $f(0) = a > a - 1 = f(b)$.

This contradicts f being order preserving.

2) Assume, BWOC, that there is an order preserving bijection f from N to $\mathsf{Q}^{\geq 0}$. Let $f(0) = a$ and $f(1) = b$. Since $0 < 1$ we know that $a < b$. Recall that $a, b \in \mathsf{Q}^{\geq 0}$ Let $c = \frac{a+b}{2}$. Since f is a bijection there exists $d \in \mathsf{N}$ such that $f(b) = d$. Note that

Since $a < c < b$ we know that $0 < d < 1$. This is a contradiction since $d \in \mathsf{N}$.

3. (30 points) Prove or disprove: If $A_1, A_2, \ldots$ are countable and disjoint then $A_1 \times A_2 \times A_3 \times \cdots$ is countable.

SOLUTION TO PROBLEM THREE

This is FALSE.

Proof Omitted