250 MIDTERM II

<u>Do not open this exam until you are told.</u> Read these instructions:

1. This is a closed book exam, though ONE sheet of notes is allowed. **No calculators, or other aids are allowed**. If you have a question during the exam, please raise your hand.
2. There are 4 problems which add up to 100 points. The exam is 1 hours 15 minutes. (You shouldn't need that much.)
3. For each question show all of your work and **write legibly**. **Clearly indicate** your answers. No credit for illegible answers.
4. Please write out the following statement: "*I pledge on my honor that I will not give or receive any unauthorized assistance on this examination.*"
5. Fill in the following:

NAME :
SIGNATURE :
SID :
SECTION NUMBER :

1. (25 points)

   (a) Let $x, y \geq 10$. There are x males and y females on the committee to revise CMSC 250. Let $1 \leq x' \leq x$ and $1 \leq y' \leq y$. The dean will choose a subcommittee of x' males and y' females. How many ways can the Dean do this?

   (b) The Dean does not want Alice (a female) and Bob (a male) to both be on the subcommittee. NOW how many ways can the Dean choose the subcommittee.

SOLUTION TO PROBLEM ONE

a) $\binom{x}{x'}\binom{y}{y'}$

b)

SOLUTION ONE:

There are three disjoint cases

Alice is on the subcommittee but Bob is not:

$\binom{x-1}{x'}\binom{y-1}{y'-1}$

Bob is on the subcommittee but Alice is not:

$\binom{x-1}{x'-1}\binom{y-1}{y'}$

Neither is on the subcommittee

$\binom{x-1}{x'}\binom{y-1}{y'}$

So the answer is

$$\binom{x-1}{x'}\binom{y-1}{y'-1} + \binom{x-1}{x'-1}\binom{y-1}{y'} + \binom{x-1}{x'}\binom{y-1}{y'}$$

This raises an interesting question: What if $x = x'$ and $y = y'$ so you MUST include Alice and Bob. To be concrete let $x = x' = 5$ and $y = y' = 7$. Then we get

$$\binom{4}{5}\binom{6}{6} + \binom{4}{4}\binom{6}{7} + \binom{4}{5}\binom{6}{7}$$

What is $\binom{4}{5}$? The number of ways to pick 5 students from 4. This is defined to be 0. If we expand it we get

$\frac{4!}{5!(-1)!}$.

Formally $(-1)!$ is infinity.

SOLUTION TWO:

There are $\binom{x}{x'}\binom{y}{y'}$ ways to form a committee. We want to SUBTRACT all of those where Alice and Bob are on the same committee. How many ways CAN Alice and Bob be on the same committee? $\binom{x-1}{x'-1}\binom{y-1}{y'-1}$. So the answer is also

$$\binom{x}{x'}\binom{y}{y'} - \binom{x-1}{x'-1}\binom{y-1}{y'-1}.$$

INTERESTING NOTE: We did the problem two different ways and got different answers. This gives a combinatorial identity!

GRADING: 15 points for part 1, 10 points for part 2.

2. (25 points) What is the coefficient of $x^{10}y^5$ in

$$(x+2y)^{15}$$

SOLUTION TO PROBLEM TWO

The number of terms that have 10 x's and 5 y's is $\binom{15}{10}$. But every time you get a y you also get a 2, so its

$$\binom{15}{10}2^5$$

GRADING:

If you got $\binom{15}{10}$ you got 10 points. If you had a another factor but it wasn't 2^5 you got 10 points

3. (25 points) Let $k, n \in \mathsf{N}$ with $3 \le k \le n$. Fill in the blanks in the following statement. Describe your reasoning. BLANK will be a function of k, n, for example BLANK could be $k + \lceil \lg n \rceil$ (it is NOT that!).

   If $A \subseteq \{1, \ldots, n\}$ and $|A| = k$ then at least BLANK subsets of A OF SIZE 3 have the same SUM.

   (NOTE the OF SIZE 3)

   Make BLANK as large as possible using the methods of this course.

   SOLUTION TO QUESTION THREE

   There are $\binom{k}{3}$ subsets of A OF SIZE 3.

   The min sum is $1 + 2 + 3 = 6$

   The max sum is $n + (n-1) + (n-2) = 3n - 3$

   SO all sums are in

$$6, 7, \ldots, 3n - 3$$

   So there are $(3n - 3) - 5 = 3n - 8$ sums.

   Hence the number of sets that have the same sum is at least

$$\left\lceil \frac{\binom{k}{3}}{3n-8} \right\rceil.$$

   GRADING:

   A correct answer got 25 points. If you had $3n - 7$ or $3n - 9$ you still got full credit if you indiated you knew min sum is 6.

   If you had the minsum is 0 instead of 6, but everything else is right, you got 20 points

   If you mixed up the denom and the num, so either $\binom{3n-8}{\binom{k}{3}}$ or something similar, you got 15 points.

   Most anything else got 0.

4. (25 points) Let $T(n)$ be defined by $T(1) = 0$

$$(\forall n \geq 1)\left[T(n) = T\left(\left\lfloor \frac{n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{2n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{11} \right\rfloor\right) + 2n\right]$$

Use constructive induction to find a constant $A \in \mathsf{N}$ such that

$$(\forall n \geq 0)[T(n) \leq An].$$

NOTE- if you do not have enough room go to the NEXT page.

ONLY USE THIS FOR PROBLEM FOUR

IF YOU TEAR THIS PAGE OUT YOU WILL LOSE 10 POINTS

SOLUTION TO PROBLEM FOUR

We do a proof that $T(n) \leq An$ and see what conditions on A we get.

Base Case: $T(1) = 0 \leq A \times 1$. No condition needed here.

IH: For all $n' < n$ $T(n') \leq An'$

IS:

By definition:

$$T(n) = T\left(\left\lfloor \frac{n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{2n}{11} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{11} \right\rfloor\right) + 2n]$$

By the IH:

$$\leq \frac{An}{11} + \frac{2An}{11} + \frac{3An}{11} + 2n$$

WANT:

$$\frac{An}{11} + \frac{2An}{11} + \frac{3An}{11} + 2n \leq An$$

$$\frac{A}{11} + \frac{2A}{11} + \frac{3A}{11} + 2 \leq A$$

$$\frac{6A}{11} + 2 \leq A$$

$$2 \leq \frac{5A}{11}$$

$$2 \times \frac{11}{5} \leq A$$

$$A = \frac{22}{5}$$

Need

$$A\alpha n + A\beta n + \gamma n \leq An$$

$$A\alpha + A\beta + \gamma \leq A$$

$$\gamma \leq A(1 - \alpha - \beta)$$

$$A \geq \frac{\gamma}{1 - \alpha - \beta}$$

So take $A = \frac{\gamma}{1-\alpha-\beta}$.

GRADING

If you did not use induction, 0 points

If you set it up correctly but get lost in the middle 15 points. (If you got 15 and appeal this you will almost surely go down to 10 since we were generous here.)

If you got the everything right but messed up some elementary algebra at the very end, you get full credit, 25 points.