Homework 6

250H

If $n_1 \equiv 5 \pmod{10}$ *and* $n_2 \equiv 10 \pmod{20}$ *then* $n_1 n_2 \equiv X \pmod{Y}$

We show how to think about the problem and then give the answer.

$n_1 \equiv 5 \pmod{10}$, so $n_1 = 10k_1 + 5$ for some k_1.

$n_2 \equiv 10 \pmod{20}$, so $n_2 = 20k_2 + 10$ for some k_2.

SO

$n_1 n_2 = 200k_1k_2 + 100k_1 + 100k_2 + 50$

AH, so

$n_1 n_2 = 100(2k_1k_2 + k_1 + k_2) + 50$

SO

$n_1 n_2 \equiv 50 \pmod{100}$.

For all $x, y \in \mathsf{Q} - \{0\}$, $x\pi + y \notin \mathsf{Q}$.

Let $x, y \in \mathsf{Q} - \{0\}$.

Assume, by way of contradiction, that $x\pi + y \in \mathsf{Q}$.

so there exists $a, b \in \mathsf{N}$ such that

$x\pi + y = \frac{a}{b}$.

Hence

$\pi = (\frac{a}{b} - y)/x$.

Since rationals are closed under $+, -, \div, \times$ we have that π is rational, which is a contradiction.

$$(\forall x, z \in \mathsf{Q})[x < z \to (\exists y \notin \mathsf{Q})[x < y < z]$$

Let $x, z \in \mathsf{Q}$ with $x < z$. Let $n \in \mathsf{N}$ be such that $\frac{\pi}{n} < z - x$. Then we have

$$x < x + \frac{\pi}{n} < z$$

By Part 1 $x + \frac{\pi}{n} \notin \mathsf{Q}$.

$$(\forall x, z \notin \mathsf{Q})(\exists y \in \mathsf{Q})[x < y < z]$$

We will assume $x, z \in (0,1)$. We leave it to the reader to adjust the proof for other cases.

Let $x = 0.x_1 x_2 \cdots$.

Let $z = 0.z_1 z_2 \cdots$.

Since $x < z$ there exists a least i such that:

$x_1 = z_1$

$x_2 = z_2$

$\vdots$

$x_{i-1} = z_{i-1}$

$x_i < z_i$.

Let

$y = x_1 \cdots x_{i-1} z_i$.

This is a finite expansion so $y \in \mathsf{Q}$.

Clearly

$x < y < z$.

Prove or disprove: $x_1 + y_1 \equiv x_2 + y_2$.

$x_1 = x_2 + km_x$
$y_1 = y_2 + km_y$
ADD these together to get:

$$x_1 + y_1 = x_2 + y_2 + k(m_x + m_y).$$

$x_1 + y_1 \equiv x_2 + y_2$.

Prove or disprove: $x_1 y_1 \equiv x_2 y_2$.

$x_1 = x_2 + km_x$

$y_1 = y_2 + km_y$

MULTIPLY these together to get:

$$x_1 y_1 = x_2 y_2 + km_x y_1 + km_y x_2 + k^2 m_x m_y = x_2 y_2 + k(m_x y_1 + m_y x_2 + km_x m_y)$$

So

$x_1 y_1 \equiv x_2 y_2$.

Prove or disprove: $x_1^{y_1} \equiv x_2^{y_2}$.

We NEED a counterexample to show that this is FALSE.

$m = 5.$

$x_1 = 2,\ x_2 = 2,\ y_1 = 7,\ y_2 = 8.$

$x_1^{x_2} = 2^2 = 4 \equiv 4 \pmod 5.$

$y_1^{y_2} = 7^8 = 5764801 \equiv 1 \pmod 5.$

AH- $1 \not\equiv 4 \pmod 5$.

Honors HW 7

What is the coefficient of x^{2021} in the Taylor Expansion of

$$\frac{1}{x^8 - x^7 - x + 1}.$$

Do by hand (NO programming) and show your work.

$$\frac{1}{x^8 - x^7 - x + 1} = \frac{1}{x-1}\frac{1}{x^7-1} = \frac{1}{1-x}\frac{1}{1-x^7}$$

$$= (1 + x + x^2 + x^3 + \cdots)(1 + x^7 + x^{14} + \cdots)$$

The coefficient of x^n is the number of ways to make n cents with 1-coins and 7-coins. We call 1-cent coins **pennies** and 7-cent coins **emilies**.

Let

$f(n)$ be the number of ways to make n cents using pennies and emilies.

$f(0) = 1$

$f(1) = f(2) = \cdots f(6) = 1$.

$f(7) = 2$: either 7 pennies or 1 emily.

$f(8) = 2$: you NEED to use 1 penny. After that you have $f(7)$.

More generally, of $n \in \mathsf{N}$ and $0 \le i \le 6$, then

$f(7n + i) = n + 1$.

$2021 = 7 * 288 + 5$.

So $f(2021) = 289$.

So the answer is 289.