BILL AND EMILY RECORD LECTURE!!!!

UNTIMED PART OF FINAL IS MONDAY May 10 9:00AM. NO DEAD CAT

FINAL IS MONDAY May 17 8:00PM-10:15PM

FILL OUT COURSE EVALS for ALL YOUR COURSES!!!

Solving One Non-Linear Recurrences

How Many Ways to Parenthesize

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Which one? Can't tell. They need to PARENTHESIZE.

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(3) (ALL THE WAYS TO DO $x_1 \boxdot x_2 \boxdot x_3$)) $\boxdot(x_4)$, so 2.
Total: 5.

Can We Make A Recurrence?

Let a_n be the number of ways to parenthesize

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Hence
$a_1 = 1$

$$(\forall n \ge 2)[a_n = a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1]$$

We Define $a_0 = 0$ to get a Cleaner Equation

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$A(x)$ is called the **Generating Function** of the sequence $a_0, a_1, \ldots$.

Use the Recurrence and the Gen Function

The recurrence only works when $n \geq 2$. Hence we look at

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (a_0 a_n + a_1 a_{n-1} + \cdots + a_n a_0) x^n$$

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Next Slide.

The Right Hand Side

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$$\sum_{n=2}^{\infty}(a_0a_n + \cdots + a_na_0)x^n = A(x)^2 - (a_0a_1 + a_1a_0)x^1 - a_0^2x^0 = A(x)^2$$

Equate LHS and RHS

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The Taylor Series for $\sqrt{1-4x}$ is:

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The Taylor series has neg coeffs but we need pos coeffs. We take

$$-\sqrt{1-4x} = \sum_{n=0}^{\infty} \frac{2}{n}\binom{2n-2}{n-1}x^n$$

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SO we have our answer!

$$a_n = \frac{1}{n} \binom{2n - 2}{n - 1}$$

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