START RECORDING

Countability

CMSC 250

Motivation

- Two toddlers want to compare their marbles to see who has more.
- They cannot count yet.

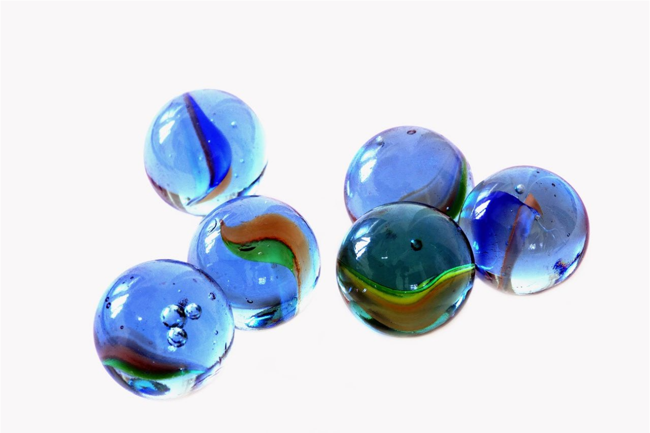
Motivation

- Two toddlers want to compare their marbles to see who has more.
- They cannot count yet.

How do they find out who has more?

Motivation

- They line them up and compare!



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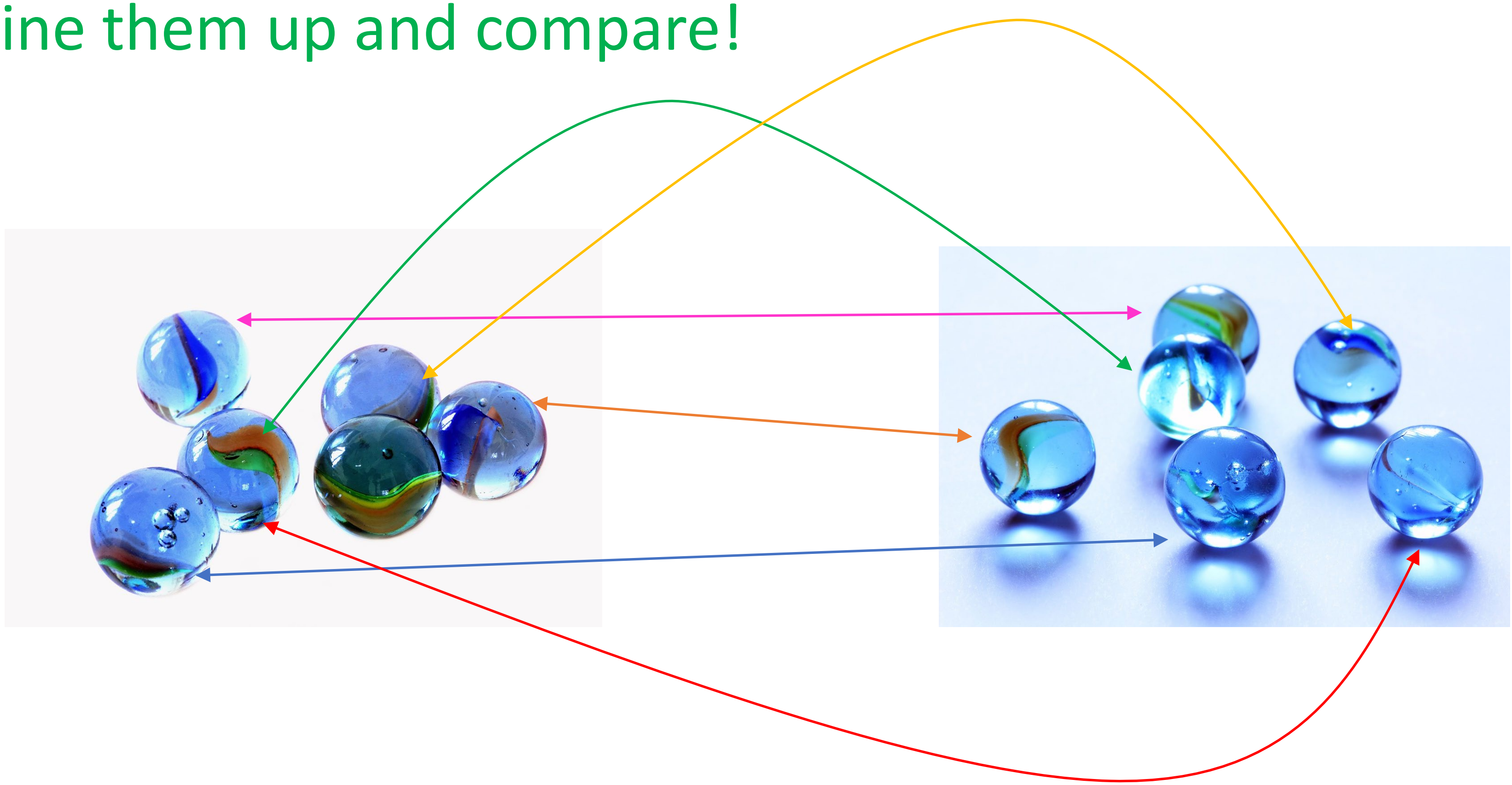

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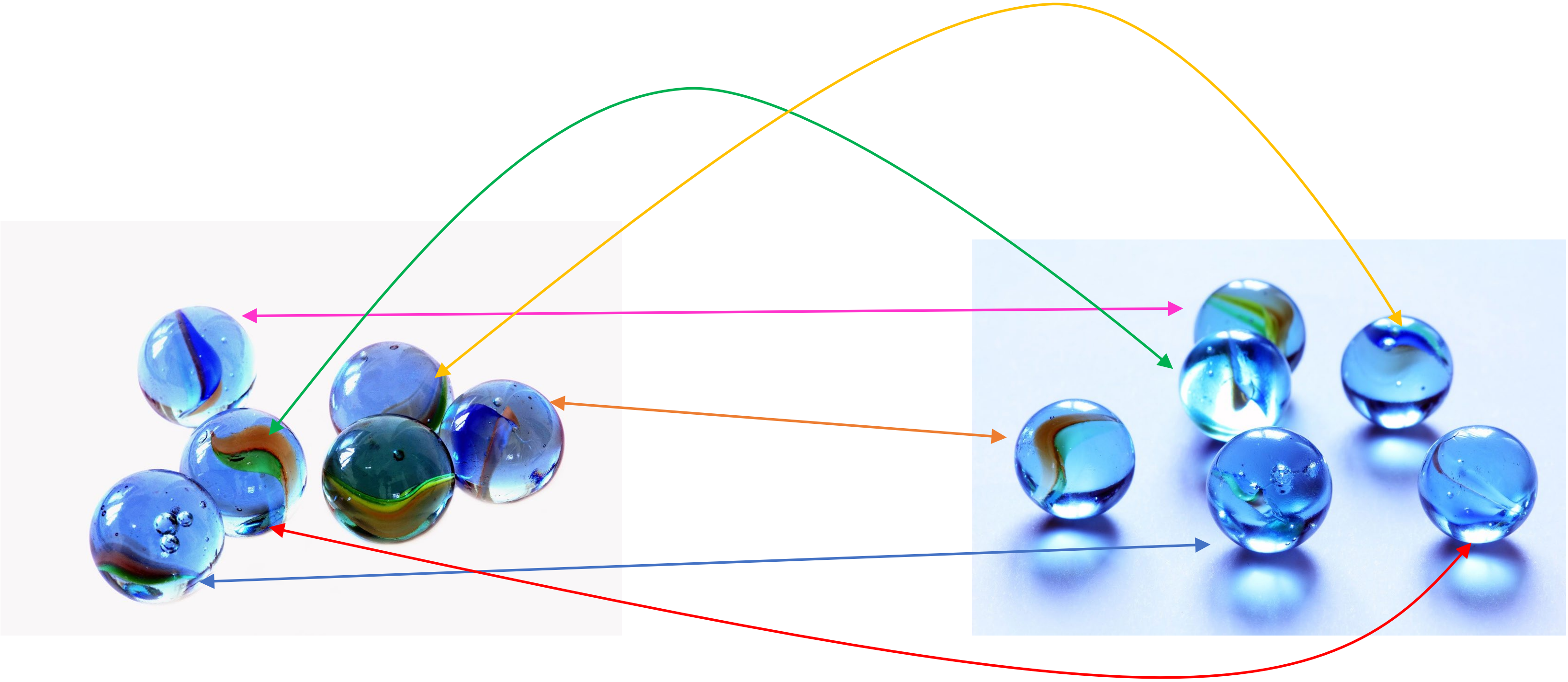
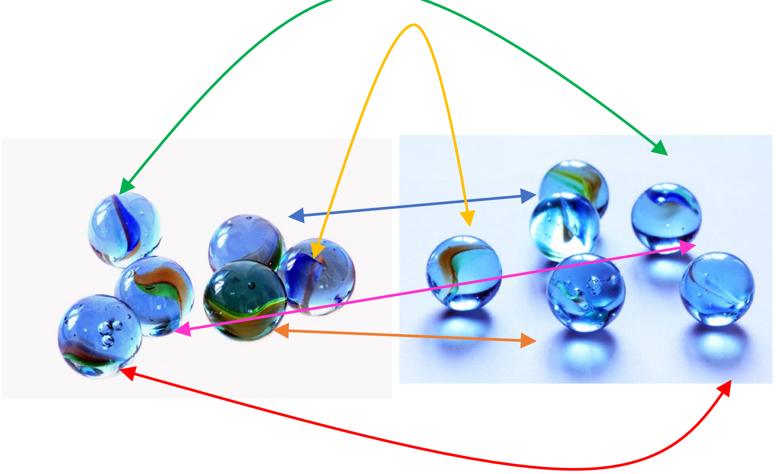


- Intuition for us: If we can find such a mapping between two (infinite) sets, we will say that they have the same (infinite) cardinality (or size).



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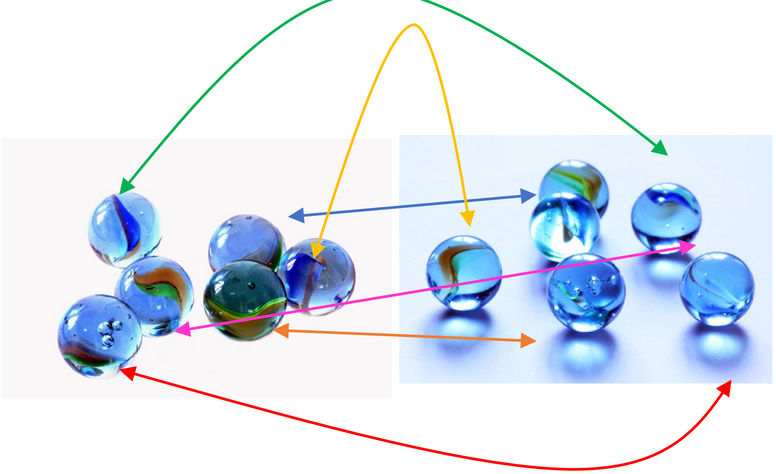
- This matching of marbles
  - Every two different marbles on left go to two different marbles on right
  - Every marble on right is matched by some marble on the left

Motivation



- This matching of marbles
  - Every two different marbles on left go to two different marbles on right
  - Every marble on right is matched by some marble on the left
- By Joav, this is a bijection!
- **WE DEFINE TWO SETS TO BE THE SAME SIZE IF THERE IS A BIJECTION BETWEEN THEM.**

Refresher on Bijections

All domains and codomains $\mathbb{R}$, unless otherwise stated

Refresher on Bijections

- Are the following functions **bijections**?

Yes

No

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1. $f(x) = |x|$

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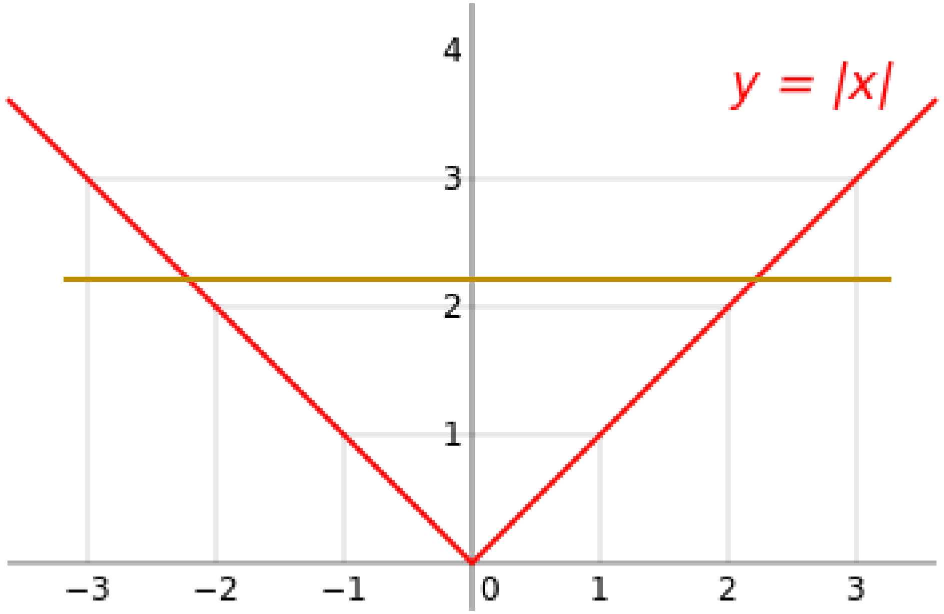
Quiz on Bijections

- Are the following functions **bijections**? Yes [✓ No]

1. $f(x) = |x|$ *No*



Non-injective!

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- Are the following functions **bijections**?

Yes

No

1. $f(x) = |x|$ *No*
2. $f(x) = a \cdot x + b,\ \ b \in \mathbb{R}, a \in \mathbb{R}^{\neq 0}$

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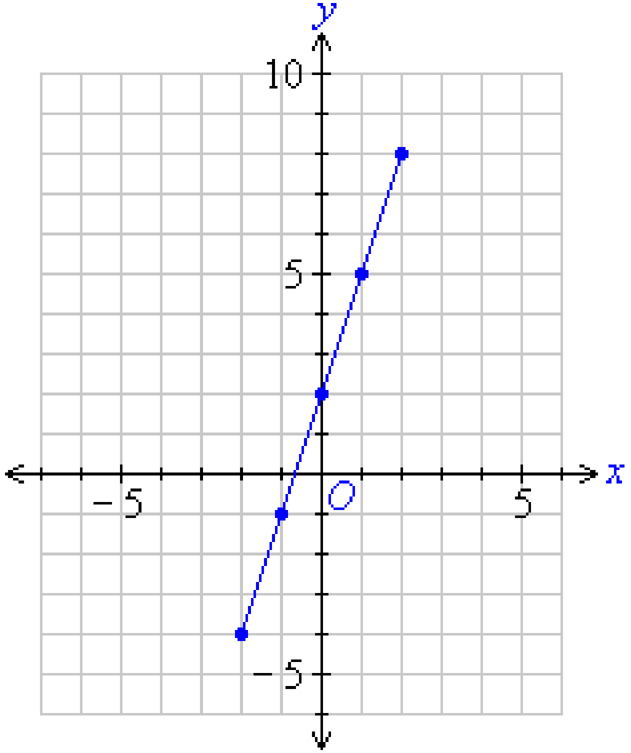

Straight line in coordinate plane

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- Are the following functions **bijections**? [✓ Yes] No

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2. $f(x) = a \cdot x + b, b \in \mathbb{R}, a \in \mathbb{R}^{\neq 0}$ *Yes*
3. $g(x) = a \cdot x^2, a > 0$

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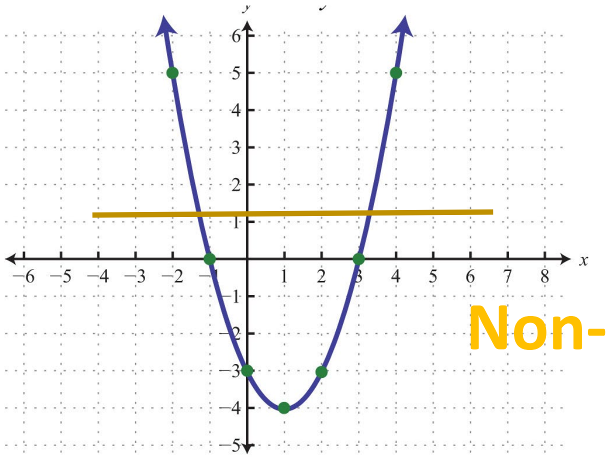

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4. $h(n) = 4n - 1, \boldsymbol{n} \in \mathbb{Z}$

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3. $g(x) = a \cdot x^2, a > 0$ No
4. $h(n) = 4n - 1, \boldsymbol{n} \in \mathbb{Z}$ No

Non-surjective! Set $h(n) = y$ and solve for n:

$$4n - 1 = y \Rightarrow n = \frac{y+1}{4}$$

There are infinitely many choices of y for which $n \notin \mathbb{Z}$!

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4. $h(n) = 4n - 1, \boldsymbol{n} \in \mathbb{Z}$ *No*
5. $h(x) = 4x - 1,$

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3. $g(x) = a \cdot x^2, a > 0$ *No*
4. $h(n) = 4n - 1, n \in \mathbb{Z}$ *No*
5. $h(x) = 4x - 1$ *Yes*

Surjective and injective! Surjective, since, if we set $h(n) = y$ and solve for n:

$$4n - 1 = y \Rightarrow n = \frac{y+1}{4}$$

For every real y, there's always a **real** solution n. Injective, since it's of the form of (2) with $a \neq 0$.

Countable Sets

- **Definition:** A set S is said to be **countable** if **there exists a bijection from a subset of $\mathbb{N}^{\geq 1}$ to S.**
  - Sometimes, this bijection is called an **enumeration.**
  - Alternatively, yet still rigorously: **If we can form some sequence out of its elements** (or, if we can **enumerate** its elements)
  - Equivalently, blending in Physics: If every one of its elements can be reached in **finite time.**

Finite Sets and Countability

- **Every finite set is countable.**

Finite Sets and Countability

- **Every finite set is countable.**
  - ***Why?***

Finite Sets and Countability

- Every finite set is countable.
  - ***Why?***
  - Suppose that S is a finite set. Since it's finite, it contains n elements, for $n \in \mathbb{N}$. This means that S can be **enumerated,** like so:

$$S = \{s_1, s_2, s_3, \dots, s_n\}$$

But this means that there exists a bijection from $\{1, 2, \dots, n\}$ to S, where $\{1, 2, \dots, n\} \subseteq \mathbb{N}$!

Infinite Sets and Countability

- Since all finite sets are countable, might as well limit ourselves to the exploration of **infinite sets** that might also be **countable**.
  - We call those "countably infinite" sets.
- Let such a set be called S. Then, to prove that it's countable, we need to find some bijection b from $\mathbb{N}^{\geq 1}$ to S.

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- Is $b: \mathbb{N}^{\geq 1} \mapsto \mathbb{N}^{\geq 1}$ such that

$$b(n) = n$$

  a bijection?

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Yes

No

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- Is $b: \mathbb{N}^{\geq 1} \mapsto \mathbb{N}^{\geq 1}$ such that

$$b(n) = n$$

a bijection?

[✓ Yes] No

Conclusion: $\mathbb{N}^{\geq 1}$ is countably infinite

Countability of $\mathbb{N}$

- Is $\mathbb{N}$ countable? (recall, $0 \in \mathbb{N}$)

Yes

No

Countability of $\mathbb{N}$

- Is $\mathbb{N}$ countable? (recall, $0 \in \mathbb{N}$)

[✓ Yes] No

- Through the bijection $f(n) = n - 1$, like so:

$$\begin{tikzcd}
\mathbb{N}^{\geq 1}: & 1, \arrow[d] & 2, \arrow[d] & 3, \arrow[d] & 4, \arrow[d] & , \ldots, & 56, \arrow[d, "f"] & \ldots \\
\mathbb{N}: & 0, & 1, & 2, & 3, & \ldots, & 55, & \ldots
\end{tikzcd}$$

Countability of Other $\mathbf{A} \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \land (x \geq 17)\}$ countable?

Yes

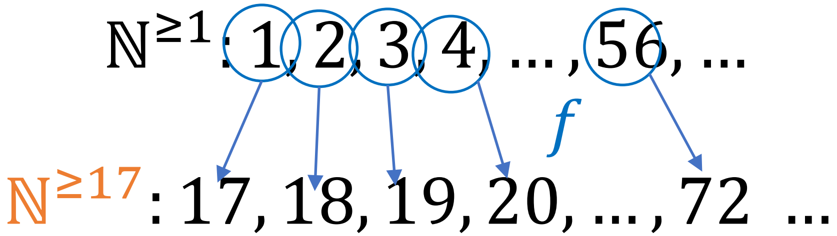
No

Countability of Other $A \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \land (x \geq 17)\}$ countable?

[✓ Yes] No

- Through the bijection $f(n) = n + 16$, like so:



$$\mathbb{N}^{\geq 1}: 1, 2, 3, 4, \ldots, 56, \ldots$$

$$f$$

$$\mathbb{N}^{\geq 17}: 17, 18, 19, 20, \ldots, 72 \quad \ldots$$

Countability of Other $A \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \land (x \equiv 0 \pmod{2})\}$ countable?

Yes

No

Countability of Other $\mathbf{A} \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \land (x \equiv 0 \ (mod\ 2)\}$ countable?

[✓ Yes] No

$\mathbb{N}^{\geq 1}$: $1, 2, 3, 4, \ldots$

$\mathbb{N}^{even}$: $0, 2, 4, 6, \ldots$

Countability of Other $\mathbf{A} \subseteq \mathbb{N}$

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\mathbb{N}^{even}: & 0, & 2, & 4, & 6, & \ldots
\end{tikzcd}$$

$$f(n) = 2(n-1)$$

Countability of $\mathbb{Z}$

- Is $\mathbb{Z}$ countable?

Yes

No

Countability of $\mathbb{Z}$

- Is $\mathbb{Z}$ countable?

[✓ Yes] No

0	1	-1	2	-2	3	-3	$\ldots$
1	2	3	4	5	6	7	

Countability of $\mathbb{Z}$

- Is $\mathbb{Z}$ countable?

[✓ Yes] No

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$f = ?$

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$$f\colon \mathbb{N} \mapsto \mathbb{Z}, f(n) = \begin{cases} \dfrac{n}{2} & if\ n\ is\ even \\ -\dfrac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

Countability of $\mathbb{Z}$

- Is $\mathbb{Z}$ countable?

[✓ Yes] No

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

1 2 3 4 5 6 7

- f is…

$$f: \mathbb{N} \mapsto \mathbb{Z}, f(n) = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ -\frac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

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1 2 3 4 5 6 7

- f is…
  - onto, since every integer is mapped to

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Countability of $\mathbb{Z}$

- Is $\mathbb{Z}$ countable?

[✓ Yes] No

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

1 2 3 4 5 6 7

- f is…
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  - 1-1, since no two naturals map to the same integer

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Countability of $\mathbb{Z}$

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[✓ Yes] No

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1	2	3	4	5	6	7	

- f is…
  - onto, since every integer is mapped to
  - 1-1, since no two naturals map to the same integer
  - So it's a bijection, and $\mathbb{Z}$ is countable!

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Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

Yes

No

Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

[✓ Yes] No

$0,$	$2,$	$-2,$	$4,$	$-4,$	$6,$	-6	$\dots$
1	2	3	4	5	6	7	

Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

[✓ Yes] No

$$0, 2, -2, 4, -4, 6, -6 \ldots$$

1 2 3 4 5 6 7

$$f(n) = \begin{cases} 0, & n = 1 \\ n, & n = 2, 4, 6, \ldots \\ -n + 1, & n = 3, 5, 7, \ldots \end{cases}$$

Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

[✓ Yes] No

$$0, 2, -2, 4, -4, 6, -6 \ldots$$

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$$f(n) = \begin{cases} 0, & n = 1 \\ n, & n = 2, 4, 6, \ldots \\ -n + 1, & n = 3, 5, 7, \ldots \end{cases}$$

Both onto and 1-1

Countability of $\mathbb{Z}^{even}$

- If f and g are bijections, then $g\left(f(x)\right) = (f \circ g)(x)$ is also a bijection

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    - Prove this at home!

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Countability of $\mathbb{Z}^{even}$

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- So, since we know that $\mathbb{Z}$ is countable…
  - i.e that there's a bijection from $\mathbb{N}^{\geq 1}$ to $\mathbb{Z}$…

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    - ***If we find a bijection*** from $\mathbb{Z}$ to $\mathbb{Z}^{even}$ …

Countability of $\mathbb{Z}^{even}$

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    - ***If we find a bijection*** from $\mathbb{Z}$ to $\mathbb{Z}^{even}$ …
      - We will have a bijection from $\mathbb{N}^{\geq 1}$ to $\mathbb{Z}^{even}$, and $\mathbb{Z}^{even}$ is, therefore, countable!

Countability of $\mathbb{Z}^{even}$

- If f and g are bijections, then $g(f(x)) = (f \circ g)(x)$ is also a bijection
  - Prove this at home!
- So, since we know that $\mathbb{Z}$ is countable…
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    - ***<u>If we find a bijection</u>*** from $\mathbb{Z}$ to $\mathbb{Z}^{even}$ …
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$$\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots$$

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

$$f(n) = 2 * n$$

Countability of $\mathbb{Z}^{even}$

- If f and g are bijections, then $g\big(f(x)\big) = (f \circ g)(x)$ is also a bijection
  - Prove this at home!
- So, since we know that $\mathbb{Z}$ is countable…
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$$\dots, -6, -4, -2, 0, 2, 4, 6, \dots$$
$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

$$f(n) = 2 * n$$

clearly bijective

Countability of $\mathbb{Q}^{>0}$

- Is $\mathbb{Q}^{>0}$ countable?

Yes

No

Countability of $\mathbb{Q}^{>0}$

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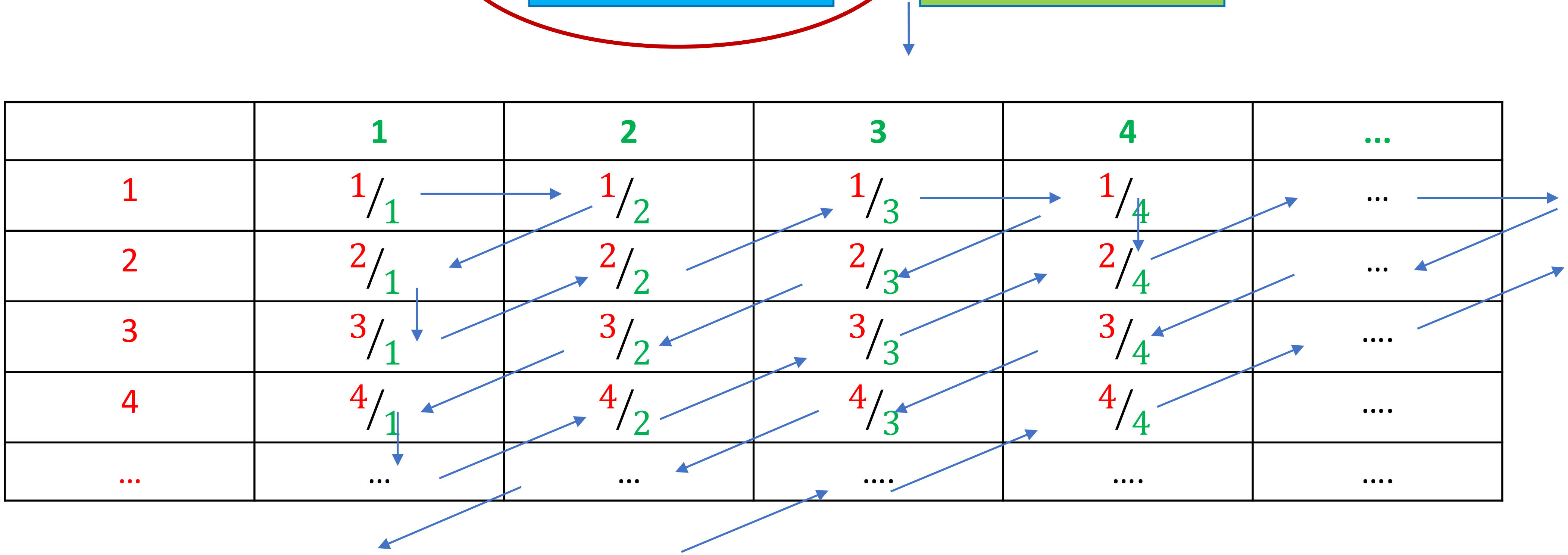
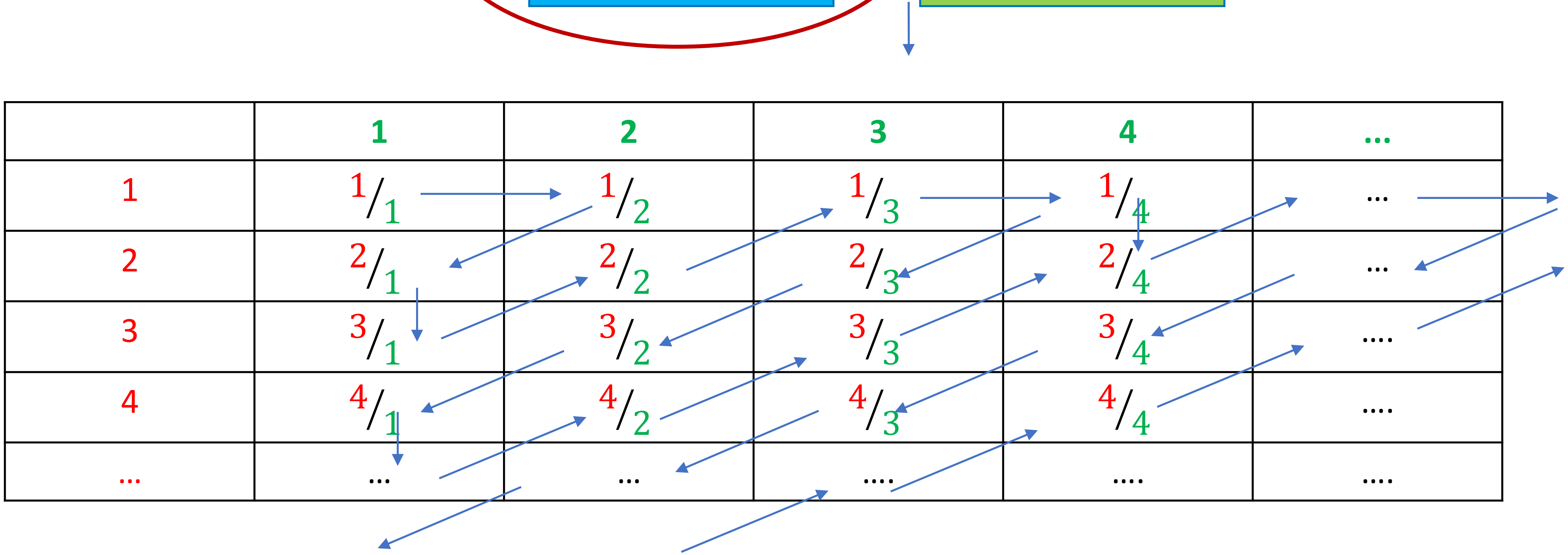


	1	**2**	**3**	**4**	**…**
1	${}^{1}/_{1}$	${}^{1}/_{2}$	${}^{1}/_{3}$	${}^{1}/_{4}$	…
2	${}^{2}/_{1}$	${}^{2}/_{2}$	${}^{2}/_{3}$	${}^{2}/_{4}$	…
3	${}^{3}/_{1}$	${}^{3}/_{2}$	${}^{3}/_{3}$	${}^{3}/_{4}$	….
4	${}^{4}/_{1}$	${}^{4}/_{2}$	${}^{4}/_{3}$	${}^{4}/_{4}$	….
…	…	…	….	….	….

Countability of $\mathbb{Q}^{>0}$

- Is $\mathbb{Q}^{>0}$ countable?

[✓ Yes] No



	1	2	3	4	…
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3	$^{3}/_{1}$	$^{3}/_{2}$	$^{3}/_{3}$	$^{3}/_{4}$	….
4	$^{4}/_{1}$	$^{4}/_{2}$	$^{4}/_{3}$	$^{4}/_{4}$	….
…	…	…	….	….	….

All strictly positive rationals are counted **exactly once** *(skipping repetitions like* $\frac{1}{1}=\frac{2}{2}=\frac{3}{3}=\cdots$*)*, so this "snaking" is a bijection

Countability of $\mathbb{Q}^{>0}$

- If you don't like the proof involving this ”snaking” pattern, ProofWiki has 4 (!) different proofs here: http://www.homeschoolmath.net/teaching/rational-numbers-countable.php
  - (1) tries to prove the “snaking” pattern in a way that I don't find very rigorous
  - 2, 3, 4 assume other facts that we won't prove today, but are easy to prove
    - E.g the cartesian product of countable sets is also countable, or the union of countable sets is also a countable set!

Some Theorems on Countability

- Suppose A is a countable set and $e \notin A$. Is $A \cup \{e\}$ countable?

Yes

No

Unknown to science

Some Theorems on Countability

- Suppose A is a countable set and $e \notin A$. Is $A \cup \{e\}$ countable?

[✓ Yes] No Unknown to science

- Suppose a_1, a_2, a_3,… is an enumeration of A.
- We then define a new enumeration b of $A \cup \{e\}$, like so:

$$b_n = \begin{cases} e, & n = 1 \\ a_{n-1}, & n \geq 2 \end{cases}$$

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Pretty much like in the case of $\mathbb{N}$, we just "move one index over"!

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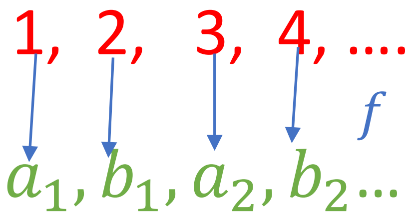
- For simplicity, assume A and B are countably infinite.



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- For simplicity, assume A and B are countably infinite.

$$\begin{tikzcd}
1, \arrow[d] & 2, \arrow[d] & 3, \arrow[d] & 4, \arrow[d, "f"] & \dots. \\
a_1, & b_1, & a_2, & b_2 & \dots
\end{tikzcd}$$

$$f(n) = \begin{cases} a_{(n+1)/2}, & n\ odd \\ b_{n/2}, & n\ even \end{cases}$$

What if A or B (or both) finite?

- Caveat: the previous will **not** work if A or B end before the other ends.
  - Because some a_i, b_i might not exist.
- We leave it to you to iron out the details of what happens then.

Note: $A \cup B \cup C$ countable

- If A, B, C are countable, so is $A \cup B \cup C$.
  - Since A, B are countable, $(A \cup \ B) = S_1$ is countable
  - $(A \cup \ B) \cup \ C = S_1 \cup \ C$. Since S_1, C are countable, $S_1 \cup \ C = (A \cup \ B) \cup \ C$ is countable.

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- Generally,

$A_1, A_2, A_3, \ldots A_n$ countable $\Rightarrow \bigcup_{i=1}^{n} A_i$ countable

(Countable union of countable sets is countable)

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- Generally,

$A_1, A_2, A_3, \ldots A_{n,} A_{n+1}, \ldots$ countable $\Rightarrow \bigcup_{i=1}^{+\infty} A_i$ countable

(Countable union of countable sets is countable)

Proof on next slide!

Countable Union of Countable Sets Countable

- Here's a proof that uses the snaking patern.
- Suppose $A_i = \{a_{i_j}, j \in \mathbb{N}\}$. Then, we can arrange the elements of the A_i'th set in the i^{th} row of a 2D matrix:

Countable Union of Countable Sets Countable

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	1st element	2nd element	3rd element	4th element	…
A_1	a_{1_1}	a_{1_2}	a_{1_3}	a_{1_4}	…
A_2	a_{2_1}	a_{2_2}	a_{2_3}	a_{2_4}	…
A_3	a_{3_1}	a_{3_2}	a_{3_3}	a_{3_4}	…
A_4	a_{4_1}	a_{4_2}	a_{4_3}	a_{4_4}	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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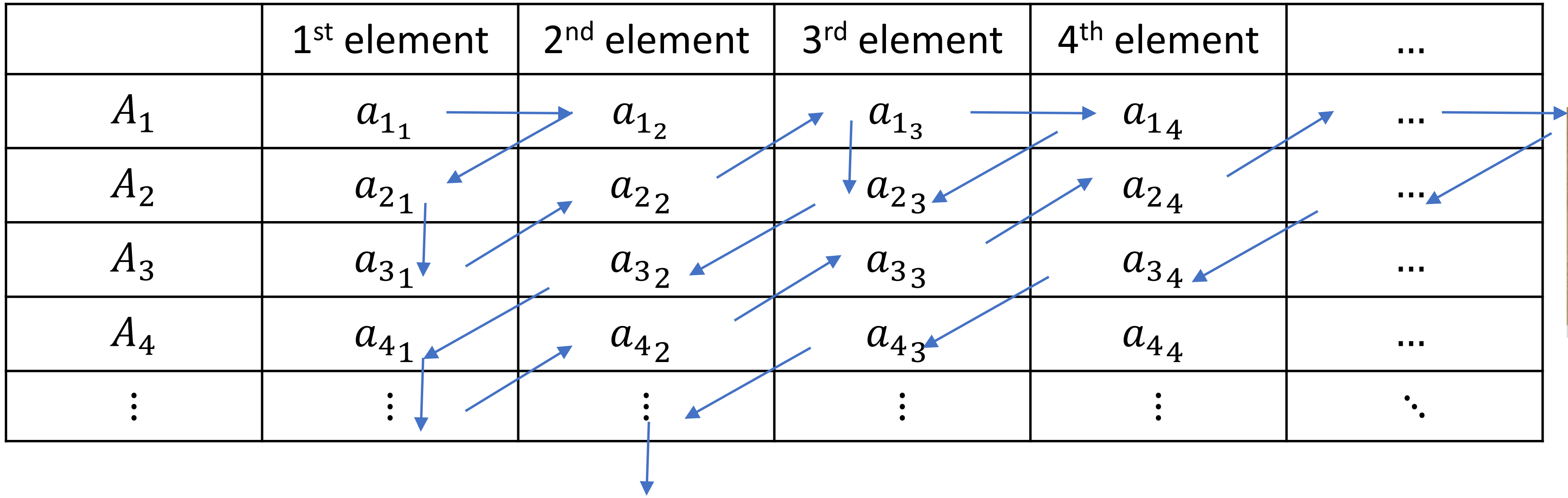
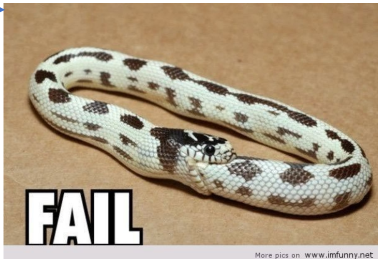


	1st element	2nd element	3rd element	4th element	…
A_1	a_{1_1}	a_{1_2}	a_{1_3}	a_{1_4}	…
A_2	a_{2_1}	a_{2_2}	a_{2_3}	a_{2_4}	…
A_3	a_{3_1}	a_{3_2}	a_{3_3}	a_{3_4}	…
A_4	a_{4_1}	a_{4_2}	a_{4_3}	a_{4_4}	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$



Snake 'em!

Some Theorems on Countability

- Suppose A and B are countable sets. Is $A \times B$ countable?

Yes

No

Unknown to science

Some Theorems on Countability

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[✓ Yes]

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- Proof is exactly the same as the proof that $\mathbb{Q}^{>0}$ is countable!

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	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	**…**
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3)	(a_1, b_4)	…
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3)	(a_2, b_4)	…
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3)	(a_3, b_4)	….

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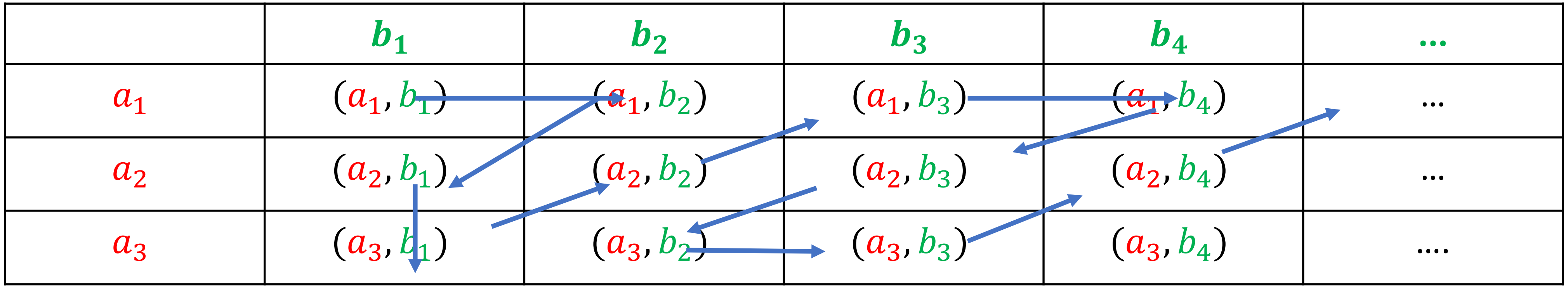

	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_4}$	**…**
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3)	(a_1, b_4)	…
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3)	(a_2, b_4)	…
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3)	(a_3, b_4)	….

Countability of $\mathbb{R}$

- Is $\mathbb{R}$ countable?

Yes

No

Unknown to science

Countability of $\mathbb{R}$

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Countability of $\mathbb{R}$

- Is $\mathbb{R}$ countable?

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- Cantor's famous **diagonal argument!**

Countability of $\mathbb{R}$

- Is $\mathbb{R}$ countable?

Yes [✓ No] Unknown to science

- Cantor's famous **diagonal argument!**
- The argument actually proves that the interval [0,1] is uncountable, but the result generalizes to the entirety of $\mathbb{R}$
  - Wait a few lectures to see why this is true.

Cantor's Diagonal Argument

- **Proof by contradiction:** Suppose that $[0, 1]$ is countable. Then, there exists some bijection from $\mathbb{N}^{\geq 1}$ to $[0, 1]$, i.e the reals can be enumerated in a sequence:

1. 0.28422856233…..
2. 0.28422856232…..
3. 0.28422856231…..

……………………………

……………………………

$n.$ 0.28422855001……

Cantor’s Diagonal Argument

$$r_1 = 0.28422856233\ldots$$
$$r_2 = 0.28422856232\ldots$$
$$r_3 = 0.28422856231\ldots$$

⋮ = ...

⋮ = ...

$$r_n = 0.2842285500\ldots$$

- Let’s create the real number $r = 0.a_1a_2a_3\ldots a_n\ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \leq r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

Cantor’s Diagonal Argument

$r_1 = 0.28422856233\ldots$
$r_2 = 0.28422856232\ldots$
$r_3 = 0.28422856231\ldots$
$\vdots$ = ……………………………
$\vdots$ = ……………………………
$r_n = 0.2842285500\ldots$

- Let’s create the real number $r = 0.a_1a_2a_3 \ldots a_n \ldots$ where

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Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In our case, $r = 0.395\ldots$

Cantor’s Diagonal Argument

$$r_1 = 0.\mathbf{2}8422856233\ldots$$
$$r_2 = 0.2\mathbf{8}422856232\ldots$$
$$r_3 = 0.28\mathbf{4}22856231\ldots$$

$\vdots$ = ...

$\vdots$ = ...

$$r_n = 0.2842285500\ldots$$

- Bill claims that $r = 0.395\ldots$ is the 17th real in the list.

Cantor’s Diagonal Argument

$$r_1 = 0.28422856233\ldots$$
$$r_2 = 0.28422856232\ldots$$
$$r_3 = 0.28422856231\ldots$$

$\vdots$ = ……………………………

$\vdots$ = ……………………………

$$r_n = 0.2842285500\ldots$$

- Bill claims that $r = 0.395\ldots$ is the 17th real in the list.
- But this cannot be true, since our real number was constructed such that it differs from the 17th real in the 17th decimal digit!

Cantor’s Diagonal Argument

$$\begin{aligned}
r_1 &= 0.28422856233 \ldots \\
r_2 &= 0.28422856232 \ldots \\
r_3 &= 0.28422856231 \ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
r_n &= 0.2842285500 \ldots
\end{aligned}$$

- Bill claims that $r = 0.395 \ldots$ is the 17th real in the list.
- But this cannot be true, since our real number was constructed such that it differs from the 17th real in the 17th decimal digit!
- Generally speaking, r will differ from the i^{th} real in the i^{th} digit!
  - So we can’t find an $k \in \mathbb{N}$ such that $0.395 \ldots = r_k$.
  - **Contradiction, since we assumed we can enumerate all reals in [0,1].**

Is $|\mathbb{N}| < |\mathbb{R}|$?

- Of course!
- But how can we say this rigorously?
- **Defn:** $|A| \leq |B|$ if there is an **injection** from A into B
- **Defn:** $|A| < |B|$ if there is an **injection** from A into B but there is **no surjection** from A into B!
  - **Advice**: Replace injection with “1-1 mapping” and surjection with “onto”

More Theorems on Countability

- Is the set of all functions $f: \mathbb{N} \mapsto \mathbb{N}$ countable?

Yes

No

Unknown to science

More Theorems on Countability

- Is the set of all functions $f: \mathbb{N} \mapsto \mathbb{N}$ countable?

Yes [✓ No] Unknown to science

- Cantorian proof on next slide

$\{f \mid f : \mathbb{N} \mapsto \mathbb{N}\}$ Uncountable

- Assume that the set is countable. Then, all functions from $\mathbb{N}$ to $\mathbb{N}$ can be enumerated:

$$f_1, f_2, f_3, \ldots$$

- Construct the function $g(x) = f_x(x) + 1$. g, when given input i, is different from f_i when also given input i. So there is no $k \in \mathbb{N}$ such that $f_k = g$. Contradiction. Therefore, $\{f \mid f : \mathbb{N} \mapsto \mathbb{N}\}$ uncountable.

More Theorems on Countability

- Is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ countable?

Yes

No

Unknown to science

More Theorems on Countability

- Is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ countable?

Yes [✓ No] Unknown to science

- Proof: $f: \mathbb{R} \mapsto \mathbb{R} \times \mathbb{R}$ such that $f(x) = (x, 1)$ is an **injection** **(1-1)**
  - Hence, $\mathbb{R} \times \mathbb{R}$ is **at least as big** as $\mathbb{R}$, and $\mathbb{R}$ is uncountable.
  - So , $\mathbb{R} \times \mathbb{R}$ is uncountable.

More Theorems on Countability

- Is $\mathbb{C}$ (set of complex numbers) countable?

Yes

No

Unknown to science

More Theorems on Countability

- Is $\mathbb{C}$ (set of complex numbers) countable?

Yes [✓ No] Unknown to science

- Remember: complex numbers defined as $a + b \cdot i$ for $a, b \in \mathbb{R}$.
  - $f: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$ such that $f\big((a, b)\big) = a + b \cdot i$ is a bijection from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$
  - But we know that $\mathbb{R} \times \mathbb{R}$ is uncountable. Therefore, $\mathbb{C}$ is uncountable.

More Theorems on Countability

- Let A be any uncountable set. Is there any $B \subseteq A$ that is countable?

Yes

No

Unknown to science

More Theorems on Countability

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- Consider: $[0,1]$ and $\left\{\frac{1}{x} \mid x \in \mathbb{N}^{\geq 1}\right\} \subseteq [0,1]$

More Theorems on Countability

- Let A be any uncountable set. Is there any $B \subseteq A$ that is countable?

[✓ Yes] No Unknown to science

- Consider: $[0,1]$ and $\left\{\boxed{\frac{1}{x}} \mid x \in \mathbb{N}^{\geq 1}\right\} \subseteq [0,1]$

All these are
positive rationals!

More Theorems on Countability

- Let $\{0, 1\}^{\infty}$ be the set of infinite sequences consisting only of 0s and 1s
  - Is it countable?

Yes

No

Unknown to science

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- Cantor-like proof in next slide!

The Set of Infinite Bit-strings is Uncountable

- Assume that the set is countable, then the strings can be enumerated:

1: 000111010101010…

2: 0101011110001101…

…

n: 010101000011100…

- Construct bit-string s which differs from the i^{th} string in the list in the i^{th} digit.
- Since this string is not in the list, we can't enumerate them all. Contradiction.

More Theorems on Countability

- Let A_1, A_2, A_3 … be an infinite sequence of countable sets.
- Is $A_1 \times A_2 \times A_3 \times \cdots$ countable?

Yes

No

Unknown
to science

More Theorems on Countability

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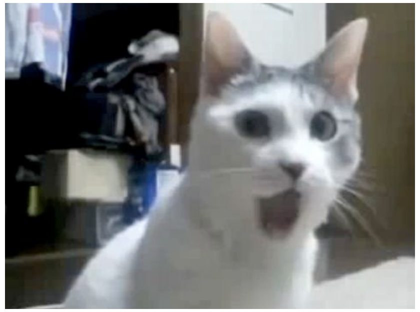

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Set of **Infinite** Cartesian Product of **Countable Sets** is **Uncountable**

- Notation: $a_i = \{a_{i_1}, a_{i_2}, a_{i_3}, \ldots\}$
- Suppose that the set is countable. Then, enumeration:

$$
\begin{gathered}
\left(a_{1_1}, a_{1_2}, a_{1_3}, a_{1_4}, \ldots\right), \\
\left(a_{2_1}, a_{2_2}, a_{2_3}, a_{2_4}, \ldots\right), \\
\left(a_{3_1}, a_{3_2}, a_{3_3}, a_{3_4}, \ldots\right), \\
\ldots
\end{gathered}
$$

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Construct infinite tuple $(a_{1_{x_1}}, a_{2_{x_2}}, a_{3_{x_2}}, \dots)$ such that x_i is an element of A_i different from the element used in the ith position of the ith tuple!

- This tuple cannot be in the list, etc etc etc

More Theorems on Countability

- Is $\mathcal{P}(\mathbb{N})$ (the powerset of the naturals) countable?

Yes

No

Unknown to science

More Theorems on Countability

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Powerset of Naturals Uncountable

- Assume that $\mathcal{P}(\mathbb{N})$ is countable. This means that we can arrange all of the subsets of $\mathbb{N}$ in a sequence: $S_1, S_2, \ldots$

- Let $A = \{i \in \mathbb{N} \mid i \notin S_i\} \subseteq \mathbb{N}$

- By construction, A cannot be in the list of subsets.

- Contradiction. So $\mathcal{P}(\mathbb{N})$ uncountable.

Infinite Number of Infinities

- We just showed that $\mathbb{N} < \mathcal{P}(\mathbb{N})$
- Similar proof: for any set A, $A < \mathcal{P}(A)$

$$\mathbb{N} < \mathcal{P}(\mathbb{N}) < \mathcal{P}\big(\mathcal{P}(\mathbb{N})\big) < \mathcal{P}\Big(\mathcal{P}\big(\mathcal{P}(\mathbb{N})\big)\Big) < \cdots$$

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- How many levels of infinity are there?

Countably many

Uncountably many ($\mathbb{R}$)

More than $\mathbb{R}$

42

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STOP RECORDING