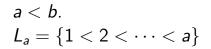
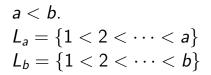
Duplicator-Spoiler Games

a < *b*.





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We will call SPOIL S and DUP D to fit on slides.

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- If this map is order preserving D wins, else S wins. Bill plays a student $(L_3, L_4, 2)$, $(L_3, L_4, 3)$

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- We assume both players play perfectly.
- We want k such that
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 - D beats S in the $(L_a, L_b, k-1)$ game.

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- GENERALLY: Who wins (L_a, L_b, k) .

Can use any orderings L, L'

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- If *L* is an ordering then L^* is that ordering backwards. **Play a student** \mathbb{N} and \mathbb{Z} with 1 move, 2 moves

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- D wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k 1)$, S wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k)$.

Breakout Rooms!

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- D wins $(\mathbb{N} + \mathbb{Z}, \mathbb{N}, k 1)$, S wins $(\mathbb{N} + \mathbb{Z}, \mathbb{N}, k)$.

A Notion of L, L' being Similar

Let L and L' be two linear orderings.

Definition

If D wins the k-round DS-game on L, L' then L, L' are k-game equivalent (denoted $L \equiv_k^G L'$).

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If $Q \in \{\exists, \forall\}$ then

$$\operatorname{qd}((Qx_1)[\phi(x_1,\ldots,x_n)] = \operatorname{qd}(\phi_1(x_1,\ldots,x_n)) + 1$$

$(\forall x)(\forall z)[x < z \rightarrow (\exists y)[x < y < x]]$

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 $\operatorname{qd}((\forall x)(\forall z)[x < z \to (\exists y)[x < y < x]]) = 2 + 1 = 3$

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Definition

L and L' are k-truth-equiv $(L \equiv_k^T L')$

 $(\forall \phi, qd(\phi) \leq k)[L \models \phi \text{ iff } L' \models \phi.$

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• $L \equiv_k^T L'$ • $L \equiv_k^G L'$

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- Density *cannot* be expressed with qd 2. (Proof: Z≡^G₂ Q so Z≡^T₂ Q).
- Well foundedness cannot be expressed in first order at all! (Proof: (∀n)[N + Z≡^G_nN).
- Upshot: Questions about expressability become questions about games.