# $e^2$ is Irrational

# Begin the Proof

Assume $e^2$ is rational. So $(\exists a, b \in \mathbb{N})$ such that $e^2 = \frac{a}{b}$.
Let $n \in \mathbb{N}$ be named later. It will be even.
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**Some Things Change** We do not have that either side is in $\mathbb{N}$.

**Plan** We prove that $n!be$ is just a **wee bit bigger than a $\mathbb{N}$** and that $n!ae^{-1}$ is **just a wee bit smaller than a $\mathbb{N}$.** But they are equal! This will be our contradiction.

# Lets Look at *ben*!

From the proof that $e$ is irrational we have $C_1 \in \mathbb{N}$ such that

$$bn!e = b\left(C_1 + \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\right).$$

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We also got that the sum is $\sim \frac{1}{n-1}$. Hence

$$bC_1 \le bn!e \le bC + \frac{b}{n-1}$$

We take $n$ large enough so that $\frac{b}{n-1} < 1$. Hence there exists $D_1 = bC_1 \in \mathbb{N}$ and $0 < \delta_1 < \frac{1}{10}$.
$bn!e = D_1 + \delta_1$.

# Lets Look at $ae^{-1}n!$

We take $n$ even.

$$an!e^{-1} = an!\left(\left(1-\frac{1}{1!}+\frac{1}{2!}\pm\cdots+\frac{1}{n!}\right)+\left(-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}\pm\cdots\right)\right)$$

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$$= a\left(\left(n! - \frac{n!}{1!} + \frac{n!}{2!} \pm \cdots + \frac{n!}{n!}\right)+\left(-\frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} \pm \cdots\right)\right).$$

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$$an!e = a\left(C_2 - \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} \pm \cdots\right).$$

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Take $n$ large enough so that $0<\frac{a}{n+1}<1$.

$an!e^{-1}=aC_2-\frac{1}{n+1}$.
Hence there exists $D_2=aC_2\in\mathbb{N}$ and $0<\delta_2<\frac{1}{10}$ such that

$$an!e^{-1}=D_2-\delta_2.$$

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Since $n!be = n!ae^{-1}$ this is a contradiction.