e is Irrational

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(4) Pays out continuously? You have e bills.

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This is the one we will use:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

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(5) Cantor proved that most numbers are transcendental in 1874, and gave a method for constructing some.

Proof that e is Irrational

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Hence $2 < e < 3$ so $e \notin \mathbb{N}$.

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Assume e is rational. So $\exists a, b \in \mathbb{N}$ such that $e = \frac{a}{b}$.
Let $n \in \mathbb{N}$ be named later.
$eb = a$, so $bn!e = n!a \in \mathbb{N}$.

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$$bn!e = bn!\left(\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)+\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots\right)\right)$$

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$$bn!e = b\left(C + \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\right) \in \mathbb{N}.$$

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Lets Look at that Series

$$\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots$$

For large n we can approximate it very well by

$$\frac{1}{n} + \frac{1}{n^2} + \cdots$$

$$= \frac{1}{n}\left(1 + \frac{1}{n} + \frac{1}{n^2} + \cdots +\right) = \frac{1}{n}\frac{1}{1-(1/n)} = \frac{1}{n-1}$$

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Take n big enough and this cannot happen. Contradiction!