Induction Review

250H

Weak Induction

- 1 base case

Strong Induction

- 1 or more base case

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- Inductive step: Prove the statement true when $n = n+1$ case. YOU MUST USE YOUR HYPOTHESIS IN THIS PROOF.

Strong Induction

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- Inductive Hypothesis: For some base $\leq i \leq n$, what we are trying to prove is true
- Inductive step: Prove the statement true when $n = n+1$ case. You might have to take several steps back in the proof. YOU MUST USE YOUR HYPOTHESIS IN THIS PROOF.

Weak Induction Example: Prove that a^4-1 is divisible by 16 for all odd integers a.

Proof by Induction: First consider positive integers.

Base Case: Let $a = 1$, Then $a^4 - 1 = 1^4 - 1 = 0 \equiv 0 \; mod \; 16$

Thus our base cases hold.

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Inductive Hypothesis: Assume that for some odd integer $a >= 1$,

$a^4 - 1$ is divisible by 16.

Weak Induction Example: Prove that a^4-1 is divisible by 16 for all odd integers a.

Inductive Step: We want to look at the next odd integer.

So we will look at $a = a + 2$.

Then, we have $(a+2)^4 - 1 \equiv 15 + 32a + 24a^2 + 8a^3 + a^4 \mod 16$

$$\equiv -1 + 8a^2 + 8a^3 + a^4 \mod 16$$

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By our inductive hypothesis,

$$\equiv 8a^2(1 + a) \ mod\ 16 .$$

Note a is odd, so $1 + a$ is a even number call it $2k$ where $k \in Z$. So,

$$\equiv 8a^2(2k) \ mod\ 16 .$$

$$\equiv 0 \ mod\ 16 .$$

Doing the negative integers will follow this same form. ☽

Strong Induction Example:

Given $a_n = \begin{cases} 2 & n = 1, 2 \\ 6 & n = 3 \\ 3a_{n-3} & n > 3 \end{cases}$, prove that $a_n = 2 \cdot 3^{\lfloor \frac{n}{3} \rfloor}$ for all positive integers n.

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Proof by induction:

Base Cases: Consider a_1, a_2, and a_3,

$$a_1 = 2(3^{\lfloor \frac{1}{3} \rfloor}) = 2(3^0) = 2$$

$$a_2 = 2(3^{\lfloor \frac{2}{3} \rfloor}) = 2(3^0) = 2$$

$$a_3 = 2(3^{\lfloor \frac{3}{3} \rfloor}) = 2(3^1) = 6$$

So, our base cases hold.

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Inductive Hypothesis: Assume for $k \geq 3$ that $a_i = 2(3^{\lfloor \frac{i}{3} \rfloor})$ for all integers with $1 \leq i \leq k$.

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k+1 step:

$$a_{k+1} = 3a_{k-2}$$

. Since $k + 1 \geq 4$ and from our inductive hypothesis,

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$$a_{k+1} = 2(3^{\left\lfloor \frac{k-2+3}{3} \right\rfloor + 1})$$

$$a_{k+1} = 2(3^{\left\lfloor \frac{k-2+3}{3} \right\rfloor})$$

$$a_{k+1} = 2(3^{\left\lfloor \frac{k+1}{3} \right\rfloor})$$

Therefore by Principle of Mathematical Induction, our formula holds. □

Use strong induction to prove that $\sqrt{2}$ is irrational. [Hint: Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b]

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Inductive Hypothesis: Assume that $\sqrt{2} \neq n/b$ for any positive integer b

for $1 \leq k \leq n$.

Inductive Step: For the sake of contradiction, assume $\sqrt{2} = (n+1)/b$

for some positive integer b. Then,

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Then, $(n+1)^2$ and $(n+1)$ are even. So $(n+1)$ can be written as

$2k$ where $k \in Z$. So,

$$2b^2 = (2k)^2$$

$$2b^2 = 4k^2$$

$$b^2 = 2k^2$$

$$b^2 = 2k^2$$

Thus, b^2 is even and can be written as $2j$ where $j \in Z$.

So,

$$\sqrt{2} = (n + 1)/b$$

$$\sqrt{2} = 2k/2j$$

$$b^2 = 2k^2$$

Thus, b^2 is even and can be written as $2j$ where $j \in Z$.

So,

$$\sqrt{2} = (n+1)/b$$

$$\sqrt{2} = 2k/2j$$

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But, $k \leq n$, so this contradicts the inductive hypothesis.

So, $\sqrt{2}$ is irrational. ☽