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k-nomial Theorem and Pascal's Triangle

CMSC 250

The Binomial Theorem and Some Computational Challenges

The Binomial Theorem

- Recall the following identities from highschool:
- $(x + y)^2 = x^2 + 2xy + y^2$
- $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x + y)^4 = x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + y^4$

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- Is there a pattern here? Can we easily generate the coefficients?
  - (Some of you might already know **how**, but we doubt that you know **why**)

$(x+y)^5$

- $(x+y)^5 = (x+y) \cdot (x+y) \cdot (x+y) \cdot (x+y) \cdot (x+y)$
- What is the coefficient of x^2y^3?

$(x+y)^5$

- $(x+y)^5 = (x+y)\cdot(x+y)\cdot\ (x+y)\cdot(x+y)\cdot(x+y)$

- What is the coefficient of x^2y^3?
- There are $2^5 = 32$ terms total (many combine, eg $xxyyy,\ \ xyxyy$ are both of form x^2y^3).
- How many of those terms have 2 ‘x’s and 3 ‘y’s?

$(x+y)^5$

- $(x+y)^5 = (x+y)\cdot(x+y)\cdot\ (x+y)\cdot(x+y)\cdot(x+y)$

$xxyyy,\quad xyxyy,\quad xyyxy,\quad xyyyx,$

$yxxyy,\quad yxyxy,\quad yxyyx,$

$yyxxy,\quad yyxyx,$

$yyyxx$

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- $(x+y)^5 = (x+y)\cdot(x+y)\cdot\ (x+y)\cdot(x+y)\cdot(x+y)$

$$\left.\begin{array}{llll} xxyyy, & xyxyy, & xyyxy, & xyyyx, \\ yxxyy, & yxyxy, & yxyyx, & \\ yyxxy, & yyxyx, & & \\ yyyxx & & & \end{array}\right\} \text{All terms of form } x^2y^3$$

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- This is just choosing 2 slots out of 5 to put the 'x's in.
- There are $\binom{5}{2} = 10$ ways of doing this.

You Do This **Now**

- What is the coefficient of x^3y^4 in $(x+y)^7$?

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$$\frac{7!}{3! \cdot 4!} = \binom{7}{3}$$

$(x+y)^n$

- We now generalize the previous results:
- $(x+y)^n = (x+y)\cdot(x+y)\cdot\ldots\cdot\ (x+y)$

- Co-efficient of $x^r y^{n-r}$ = # of ways to select r 'x's from n slots = $\binom{n}{r}$

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- Binomial Theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r}\ x^r y^{n-r}$$

How to find the coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

- Approach #1: Compute **directly** via formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Problem: Large intermediary numbers, **even if $\boldsymbol{n}, \boldsymbol{r}$ and $\boldsymbol{\binom{n}{r}}$ are relatively small!**
  - Example: $\binom{20}{10} = \frac{20!}{10! \cdot 10!} = \frac{1 \times 2 \times \cdots \times 10 \times 11 \times 12 \times \cdots \times 20}{(1 \times 2 \times \cdots \times 10) \cdot (1 \times 2 \times \cdots \times 10)}$

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  - Is our computer **smart enough** to cancel out the stuff in green?
    - Not every computer is!

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    - But assuming that ours is, we still have to compute $11\ \times 12\ \times\cdots\times 20$, which is **quite large, even though the final result is small!**

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- Can we do better?
  - Yes, through Pascal's triangle!

Using Pascal's Identity and Triangle to Calculate any $\binom{n}{r}$ ***Fast***

Expanding Binomial Theorem to Trinomial, Quadrinomial, …., k-nomial

An Easy Combinatorial Identity

We will prove that

$$(\forall n, r \in \mathbb{N})\left[(r \leq n) \Rightarrow \binom{n}{r} = \binom{n}{n-r}\right]$$

in two different ways!

Another Combinatorial Identity

$$(\forall n, r \in \mathbb{N}^{\geq 1})\left[(r \leq n) \Rightarrow \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}\right]$$

Another Combinatorial Identity

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1. Algebraic proof
2. Combinatorial proof!

A Combinatorial Proof of $\binom{n}{r} = \boxed{\binom{n-1}{r-1} + \binom{n-1}{r}}$

- LHS: #ways to pick r people from a set of n people.
- RHS: Focus on one person, call him $Jason$.
  - If we pick $Jason$, then we are left with $n-1$ people to decide if we want to pick or not, from which we now have to pick $r-1$ people **(first term of RHS)**
  - OR, if we don't pick $Jason$, we are left with $n-1$ people to decide if we want to pick or not, yet still r people that we need to pick (second term of RHS).

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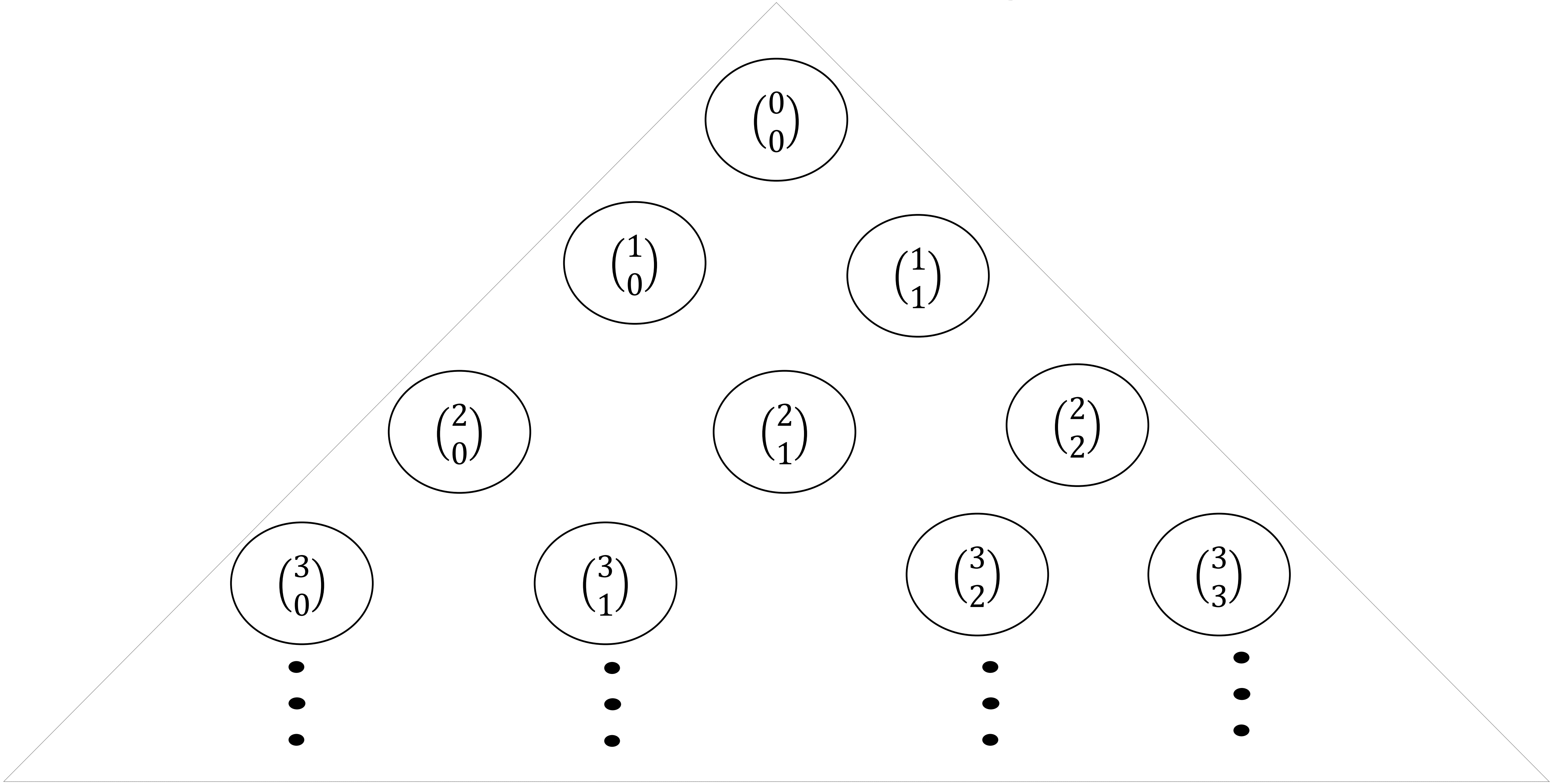
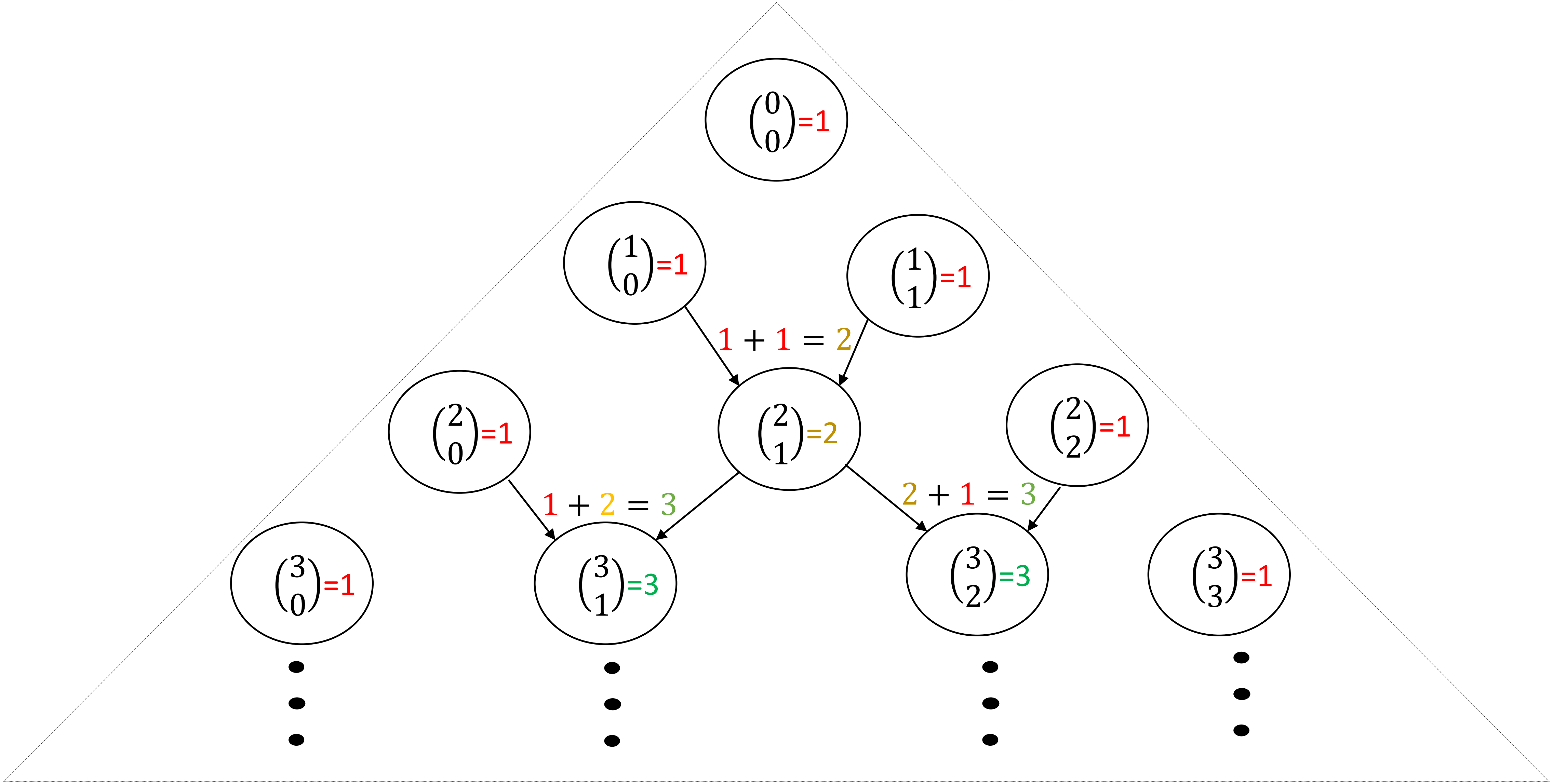
  - OR, if we don't pick $Jason$, we are left with $n-1$ people to decide if we want to pick or not, yet still r people that we need to pick (second term of RHS).
- This is a **combinatorial proof**!
- A combinatorial proof is a type of proof where we show two quantities are equal because they solve the same problem.

Pascal's Triangle



Pascal's Triangle



Upshot

- Use combinatorial identity

generate Pascal's triangle

generate binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

use in the expansion of $(x+y)^n$

Efficiency of Pascal's Triangle

- We avoid the intermediary large numbers problem
- i^{th} level of triangle gives us all coefficients $\binom{i}{0}, \binom{i}{1}, \dots, \binom{i}{i}$
- Compute the value of every node as the sum of its two parents
  - Note that the diagonal "edges" of the triangle always 1.

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- How many permutations of $x^a y^b$ are there?

$$\frac{(a + b)!}{a! \cdot b!}$$

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$$\frac{(a+b)!}{a!\cdot b!} = \frac{n!}{a!\cdot(n-a)!} = \binom{n}{a}$$

An Exercise For You To Do **Now**

- Expand $(x + y + z)^2$

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- Expand $(x + y + z)^2$

$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

Trinomial Theorem

- $(x+y+z)^5 = (x+y+z) \cdot (x+y+z) \cdot (x+y+z) \cdot (x+y+z) \cdot (x+y+z)$

- The expansion will have terms of form

$$x^a y^b z^c, \text{ where } a+b+c=5$$

- What should the coefficients be?

Trinomial Theorem

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- Once again, let's view $x^a y^b z^c$ as a string.
- #permutations of this string =

$$\frac{(a+b+c)!}{a! \cdot b! \cdot c!}$$

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- #permutations of this string =

$$\frac{(a+b+c)!}{a! \cdot b! \cdot c!} = \frac{5!}{a! \cdot b! \cdot c!}$$

Trinomial Theorem

$$(x+y+z)^n = \sum_{\substack{a+b+c=n \\ 0 \le a,b,c \le n}} \frac{n!}{a!\,b!\,c!} x^a y^b z^c$$

k-nomial Theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 0\le a_1,a_2,\ldots,a_k\le n}} \frac{n!}{a_1!\,a_2!\ldots a_k!} x_1^{a_1} x_2^{a_2} \ldots x_k{}^{a_k}$$

k-nomial Theorem

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$$\Leftrightarrow \left(\sum_{i=1}^{k} x_i\right)^n = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 0\le a_1,a_2,\ldots,a_k\le n}} \frac{n!}{\prod_{i=1}^{k} a_i!} \prod_{i=1}^{k} x_i^{a_i}$$

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