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Logical Equivalence

CMSC250

Spring 2020

Equivalences

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p	q	$p \wedge q$	$q \wedge p$
F	***F***	***F***	***F***
F	***T***	***F***	***F***
T	***F***	***F***	***F***
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- **These columns are the same!**
- **Conclusion:** $p \wedge q \equiv q \wedge p$

This symbol means
"logical equivalence"

An important equivalence

- Please fill – in the following truth table:

p	q	$p \implies q$	$(\sim p) \vee q$
F	***F***	***?***	***?***
F	***T***	***?***	***?***
T	***F***	***?***	***?***
T	***T***	***?***	***?***

$\Leftrightarrow$ vs $\equiv$

- $\Leftrightarrow$ *("if and only if")* is used to **form statements**, e.g
  - $p \Leftrightarrow (q \wedge (\sim r))$
- $\equiv$ *("logically equivalent to")* **compares two statements**, e.g
  - $(p \wedge q) \equiv (q \wedge p)$

Another important equivalence

- Let's fill in the following truth table :

a	b	$\sim (a \land b)$	$(\sim a) \lor (\sim b)$
F	*F*	?	?
F	*T*	?	?
T	*F*	?	?
T	*T*	?	?

Another important equivalence

- Let's fill in the following truth table :

$\boldsymbol{a}$	$\boldsymbol{b}$	$\sim(\boldsymbol{a} \wedge \boldsymbol{b})$	$(\sim \boldsymbol{a}) \vee (\sim \boldsymbol{b})$
F	***F***	*T*	*T*
F	***T***	*T*	*T*
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This result is known as
De Morgan's law

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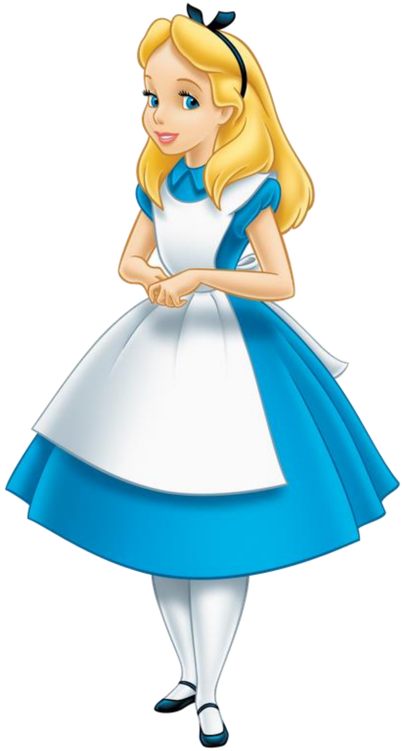
Understanding De Morgan's Law

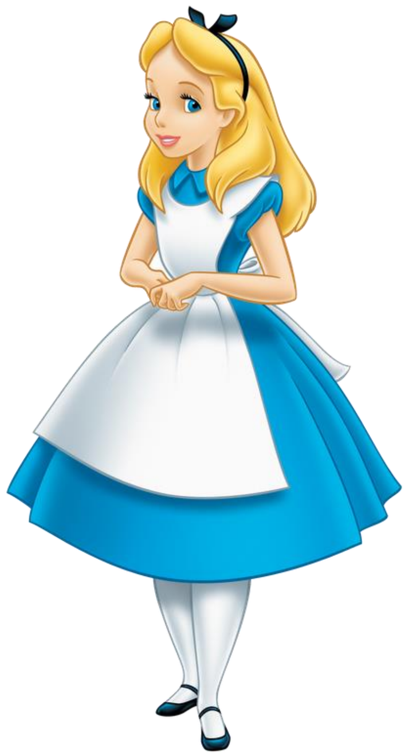
- $\sim(\text{“}\mathit{Alice\ is\ Blonde}\text{”} \wedge \text{“}\mathit{Alice\ wears\ Green\ Dress}\text{”})$: **Clearly true**



Understanding De Morgan’s Law

- $\sim($“*Alice is Blonde*” $\wedge$ “*Alice wears Green Dress*”$)$: **Clearly true**

- $(\sim$“*Alice is Blonde*”$) \vee (\sim$“*Alice wears Green Dress*”$)$:
  Also true!



De Morgan's Laws (there's two of them)

$$\sim (a \lor b) \equiv (\sim a) \land (\sim b)$$

$$\sim (a \land b) \equiv (\sim a) \lor (\sim b)$$

- Conjunctions flipped to disjunctions, and vice versa
- Negation operator ($\sim$) distributed across terms
- These laws give us our first pair of equivalent expressions!

Are these correct equivalences?

$a \Rightarrow b \equiv (\sim b) \Rightarrow (\sim a)$	$a \Rightarrow b \equiv (\sim a) \Rightarrow (\sim b)$	$a \Leftrightarrow b \equiv \big((\sim a) \vee b\big) \wedge \big((\sim b) \vee a\big)$
Left column	Middle column	Right column

Proving equivalences

- How do we prove an equivalence? (e.g $\sim(a \wedge b) \equiv (\sim a) \vee (\sim b)$)

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1. Truth tables
   - One major problem: for n variables, 2^n rows (input combinations) to enumerate!
   - Can we do better?
2. Laws of logical equivalence in a chain, one after the other!
   - We no longer have to compare 2^n input combinations to ensure that they all map to the same truth value (**T** or **F**). ☺
   - But somebody needs to code the system up!

Table of equivalences

Commutativity of binary operators	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associativity of binary operators	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity of binary operators	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation laws	$p \vee (\sim p) \equiv T$	$p \wedge (\sim p) \equiv F$
Double negation	$\sim(\sim p) \equiv p$	
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's axioms	$\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$	$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
Universal bound laws	$p \vee T \equiv T$	$p \wedge F \equiv F$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of contradictions / tautologies	$\sim F \equiv T$	$\sim T \equiv F$
Contrapositive	$(a \Rightarrow b) \equiv ((\sim b) \Rightarrow (\sim a))$	
Equivalence between biconditional and implication	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$	
Equivalence between implication and disjunction	$a \Rightarrow b \equiv \sim a \vee b$	

- This exact table will be **given to you** during **all** exams where you might need it, so that you don't have to remember some “exotic” names

Proving equivalences using laws

- Suppose we want to investigate if

$$(((a \wedge b) \vee q) \wedge (b \wedge a)) \equiv (p \vee \sim p) \wedge ((a \wedge b) \vee ((\sim r) \wedge r))$$

- How many rows would the truth table have?

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- Let's see how we could use the laws of logical equivalence to prove this equivalence
  - Important: **document the laws!**

More equivalences

- Let's prove the following equivalences **true** or **false** together.

$$a \Rightarrow b \equiv (\sim b) \Rightarrow (\sim a)$$ *(Contrapositive)*

$$a \Rightarrow b \equiv (\sim a) \Rightarrow (\sim b)$$ *(Inverse Error)*

$$a \Leftrightarrow b \equiv \big((\sim a) \vee b\big) \wedge ((\sim b) \vee a)$$

Simplifying expressions

- Large expressions can often be **simplified** using the equivalences we discussed earlier.
- Example: Let's simplify $p \land (p \lor q) \land (p \land q)$

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Here's one way
$p \land (p \lor q) \land (p \land q)$ *(Original expression)* $\equiv p \land (p \land q)$ *(How?)* $\equiv (p \land p) \land q$ *(How?)* $\equiv p \land q$ *(How?)*

Your turn, class!

- Let's simplify the following three expressions.

Left column	Middle column	Right column
$(a_1 \lor a_1) \land (a_2 \lor a_2) \land \cdots \land (a_{100} \lor a_{100}) \land (\sim a_1 \lor \sim a_1) \land (\sim a_2 \lor \sim a_2) \land \cdots \land (\sim a_{100} \lor \sim a_{100})$	$(p \land r) \lor ((p \lor s) \land (p \lor a))$	$p \land ((p \lor \sim q) \lor (\sim (\sim (z \lor \sim q))))$

Solution to 1

$$(a_1 \vee a_1) \wedge (a_2 \vee a_2) \wedge \cdots \wedge (a_{100} \vee a_{100}) \wedge (\sim a_1 \vee \sim a_1) \wedge (\sim a_2 \vee \sim a_2) \wedge \cdots \wedge (\sim a_{100} \vee \sim a_{100})$$

$\equiv a_1 \wedge a_2 \wedge \cdots \wedge (a_{100}) \wedge (\sim a_1) \wedge (\sim a_2) \wedge \cdots \wedge (\sim a_{100})$ *(Idempotence 100 times)*

$\equiv a_1 \wedge (\sim a_1) \wedge \ a_2 \wedge (\sim a_2) \wedge \cdots \wedge (a_{999}) \wedge (\sim a_{999}) \ldots \wedge (a_{100}) \wedge (\sim a_{100})$ *(Commutativity 100 times)*

$\equiv F \wedge F \wedge \cdots \wedge F \ldots \wedge F$ *(Negation 100 times)*

$\equiv F$ *(Idempotence 99 times)*

Solution to 2

$$(p \wedge r) \vee \big((p \vee s) \wedge (p \vee a)\big)$$

$$\equiv (p \wedge r) \vee (p \vee (s \wedge a\ ))$$ *(Distributivity)*

$$\equiv ((p \wedge r) \vee p) \vee (s \wedge a\ )$$ *(Associativity)*

$$\equiv (p \vee (p \wedge r)) \vee (s \wedge a\ )$$ *(Commutativity)*

$$\equiv p \vee (s \wedge a\ )$$ *(Absorption)*

Solution to 3

$$p \wedge \big((p \vee \sim q) \vee (\sim (\sim (z \vee \sim q)))\big)$$

$$\equiv p \wedge \big((p \vee \sim q) \vee (z \vee \sim q)\big)$$ *(Double Negation)*

$$\equiv p \wedge \big((p \vee z) \vee (\sim q \vee \sim q)\big)$$ *(Associativity)*

$$\equiv p \wedge \big((p \vee z) \vee \sim q\big)$$ *(Idempotence)*

$$\equiv p \wedge (p \vee (z \vee \sim q))$$ *(Associativity)*

$$\equiv p$$ *(Absorption)*

Truth Tables vs Proofs of Equivalence

- When we want to show that $\phi(x_1, x_2, \dots x_n) = \psi(x_1, x_2, \dots, x_n)$:

Truth Table		**Equivalence Proof**	
Pro	**Con**	**Pro**	**Con**
Always works	Requires 2^n ***space***	Often occupies much less than 2^n space	There are some cases where it will still take 2^n space / time
No "cleverness" needed: just build all rows mechanically	Requires 2^n ***time***	Often spends much less than 2^n time	Requires "cleverness"

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