Provably Concrete Transcendental Numbers: Liouville Numbers

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We will define a number that has **great** rational approximations and then show that all such numbers are transcendental.

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(In binary.)
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Upshot For all n there exists a, b such that $|\alpha - \frac{a}{b}| \leq \frac{1}{b^n}$.

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Notation We will call them ***L*-numbers.**

Proof that All L-numbers are Transcendental

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This is a contradiction.

The Mean Value Theorem (MVT)

MVT Let p be a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous on $[c, d]$ and differential on (c, d). Then $\exists e \in (c, d)$ such that

$$p'(e) = \frac{p(d) - p(c)}{d - c}.$$

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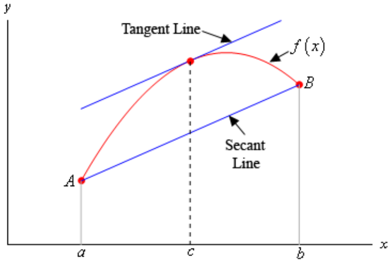
For intuition see this picture:



https://tutorial.math.lamar.edu/classes/calci/MeanValueTheorem.aspx

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Important L-numbers are all about $|\alpha - \frac{a}{b}|$ being small.

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By **Def of L-number** with param $n + r$ (we pick r later):

$$(\exists a, b, \in \mathbb{Z})\left[\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^{n+r}}\right]$$

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(2) $|p(\alpha) - p(\frac{a}{b})|$ BIG by properties of $\mathbb{Z}[x]$.
(3) By MVT and (2), $|\alpha - \frac{a}{b}|$ BIG, contradicting point (1).

Dealing with $\left|\alpha - \frac{a}{b}\right|$ Being Small

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then $\frac{a}{b} \in [\alpha - 1, \alpha + 1]$ and $\frac{a}{b} \neq \alpha_i$.

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Take r such that $A < \frac{1}{b^r}$.

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But we have

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That is the contradiction.