Induction

250H

Recursive Definitions for Functions

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- Fibonacci: {1, 1, 2, 3, 5, 8, 13, ...}
  - Recursive Definition:

$$F_n = F_{n-1} + F_{n-2}$$

  - Closed form:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example

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- Base Step: Specify the value of the function at zero
- Recursive Step: Give a rule for finding its value at an integer from its values at smaller integers

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There are two parts: the base step and the recursive step

For $n = \{0, 1, 2, 3, ...\}$

- Base step: $a^0 = 1$
- Recursive step: $a^{n+1} = a(a^n)$

Recursive Definitions for Sets and Structures

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- Sets can also be defined recursively
- We still use the basis step and the recursive step
    - Basis Step: initial collection of elements is specified
    - Recursive Step: rules for forming new elements in the set from those already known to be in the set are provided
    - (Optional) Exclusion Rule: Specifies that a recursively defined set contains nothing other than those elements specified in the basis step or generated by applications of the recursive step

Proving these things

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To prove results about recursively defined sets, we use what is called Structural Induction

Form of Structural Induction:

- Base Case: Show that the result holds for all elements specified in the basis step of the recursive definition
- Inductive Hypothesis: Assume that for some element in the set, when we apply the recursive definition, we stay in the set
- Inductive Step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for the new elements

Back to Fibonacci

$$f_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ f_{n-1} + f_{n-2} & n \geq 2 \end{cases}$$

Prove $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.

Fibonacci Induction Proof

Base Case: Let $n = 0$,

$$f_0 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^0 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$= \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0$$

Fibonacci Induction Proof

Let $n = 1$,

$$f_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^1 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right)$$

$$\frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) = 1$$

So our base cases holds.

Fibonacci Induction Proof

Inductive Hypothesis: Assume that for $0 \leq i \leq n$ where $n \geq 1$,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Fibonacci Induction Proof

Inductive Step: Consider, $f_{n+1} = f_n + f_{n-1}$. By our inductive hypothesis we have

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. So we have,

$$\frac{1}{\sqrt{5}}(\alpha^n - \beta^n + \alpha^{n-1} - \beta^{n-1})$$

$$= \frac{1}{\sqrt{5}}(\alpha^{n-1}(\alpha+1) - \beta^{n-1}(\beta+1))$$

Note that $\alpha^2 = 1 + \alpha$ and $\beta^2 = 1 + \beta$. This comes from the fact that α and β are roots of $x^2 - x - 1$. Now we have,

$$= \frac{1}{\sqrt{5}}(\alpha^{n-1}(\alpha^2) - \beta^{n-1}(\beta^2))$$

$$= \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

Example 1

Consider the following:

$$a_n = \begin{cases} 3 & n = 0 \\ 5 & n = 1 \\ 3a_{n-1}a_{n-2} + 4 & n \geq 2 \end{cases}$$

Prove $\forall n,\ a_n^2 \equiv 1 \mod 8$.

Example 1

Base Case:

Let $n = 0$. Then $a_0^2 = 3^2 \equiv 1 \mod 8$.

Let $n = 1$. Then $a_1^2 = 5^2 \equiv 1 \mod 8$.

So our base cases hold.

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Base Case:

Let $n = 0$. Then $a_0^2 = 3^2 \equiv 1 \mod 8$.

Let $n = 1$. Then $a_1^2 = 5^2 \equiv 1 \mod 8$.

So our base cases hold.

Inductive Hypothesis: Assume that for $0 \leq i \leq n$ where $n \geq 1$, $a_i^2 = 1 \mod 8$.

Example 1

Inductive Step: Consider $a_{n+1}^2 = (3a_n a_{n-1} + 4)^2$. Simplifying that we get,

$$a_{n+1}^2 = (3a_n a_{n-1} + 4)^2$$

$$= 9a_n^2 a_{n-1}^2 + 24a_n a_{n-1} + 16$$

$$\equiv 1a_n^2 a_{n-1}^2 + 0a_n a_{n-1} + 0 \mod 8$$

$$\equiv 1(1)(1) \mod 8 \textit{ by the Inductive Hypothesis}$$

$$\equiv 1 \mod 8$$

Thus, $\forall n$, $a_n^2 \equiv 1 \mod 8$.