START RECORDING

Order-Preserving Bijections

CMSC 250

$\mathbb{N}, \mathbb{Z}$: the Same, or Different?

1. There is a bijection from $\mathbb{N}$ to $\mathbb{Z}$: so same size. (AS SETS)
2. But they seem different (as ordered sets)
3. How to pin down the difference?

Order-Preserving Bijections

- **Definition:** Let A and B be ordered sets. ($A, B \subseteq \mathbb{R}$), ordering the usual ($\leq$). An **order-preserving bijection** (henceforth: **OPB**) $f: A \mapsto B$ is a bijection such that

$$(x < y) \Leftrightarrow f(x) < f(y)$$

- If A and B are two ordered sets and there exists an OPB from A to B, then we say that they are of the same **ordinality**
  - Clearly, **same ordinality implies same cardinality**.

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Note: This is an iff statement!

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$$
\begin{array}{ccccc}
0, & 1, & 2, & 3, & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0, & 2, & 4, & 6, & \ldots
\end{array}
$$

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0, 1, 2, 3, …

↓ ↓ ↓ ↓

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- Other sets with OPBs: $\mathbb{N}, \mathbb{N}^{odd}, \mathbb{N}^{\equiv(0 \; mod \; 3)}, \mathbb{N}^{\equiv i \; (mod \; j)}, \mathbb{N}^{\geq 17}, \ldots$

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- From the defn of OPBs, this means that $x < 0$.
- Contradiction, since 0 is the least natural.
- Therefore, there is no OPB from $\mathbb{N}$ to $\mathbb{Z}$.

Is $\mathbb{N} \prec \mathbb{Z}$?

- Of course!
- But how can we say this rigorously? (A, B **ordered sets**)
- **Defn:** $A \preccurlyeq B$ if there is an OP**I** (Order-Preserving **Injection**) from A into B.
- **Defn:** $A \prec B$ if there is an OP**I** from A into B but there is no OP**S** (Order-Preserving Surjection) from A into B!
  - **Advice**: Stick with injection and surjection here instead of OPONETOONE or OPONTO.

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  - **Advice**: Stick with injection and surjection here instead of OPONETOONE or OPONTO.
- Note: $A \preccurlyeq B$ is read "A is less than or equal to B" with the understanding that it applies to A, B ordered sets.

We Prove $\mathbb{N} \prec \mathbb{Z}$

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3. **Corollary:** $\mathbb{N} \prec \mathbb{Z}$
   - Follows from Theorems 1 and 2

$\mathbb{Z}, \mathbb{Q}$: Different or the Same?

1. **Theorem:** There is no OPB from $\mathbb{Z}$ to $\mathbb{Q}$.

- Proof (by contradiction): Assume there exists an OPB $f: \mathbb{Z} \mapsto \mathbb{Q}$. Let $f(0) = y_1, f(1) = y_2$. Then, f is clearly an **OPS** as well!
  - Now, finish yourselves at your desks!

$\mathbb{Z}, \mathbb{Q}$: Different or the Same?

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  - Now, finish yourselves at your desks!
- Let $x \in \mathbb{Z}$ map to $\frac{y_1+y_2}{2}$
- $(0 < 1) \Rightarrow \big(f(0) < f(1)\big) \Rightarrow (y_1 < y_2)$, since f is an OPB.
- $y_1 < \frac{y_1+y_2}{2} < y_2$ since $\frac{y_1+y_2}{2}$ arithmetic mean of y_1, y_2
- Henceforth, $0 < x < 1$
- Contradiction, since $x \in \mathbb{Z}$
- So there is no OPB from $\mathbb{Z}$ to $\mathbb{Q}$.

$\mathbb{Z} \prec \mathbb{Q}$

1. **Theorem:** There exists an OPI from $\mathbb{Z}$ to $\mathbb{Q}$.
   - Identity mapping $f(z) = z$.
2. **Theorem:** There is no OPS from $\mathbb{Z}$ to $\mathbb{Q}$.
   - Follows from proof that there is no OPB from $\mathbb{N}$ to $\mathbb{Z}$

- **Corollary:** $\mathbb{Z} \prec \mathbb{Q}$
  - Follows from Theorems 1 and 2
- Note that we now have:

$$\mathbb{N} \prec \mathbb{Z} \prec \mathbb{Q}$$

Orderings of Type $\mathbb{N}$

- Recall: The following sets are of **cardinality** $\aleph_0$:
  - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}, \ldots$
- What sets are of **ordinality** N?
  - $\mathbb{N}, \mathbb{N}^{even}, \mathbb{N}^{odd}, \mathbb{N}^{\equiv(0\ (mod\ 3)}, \mathbb{N}^{\equiv i\ \ (mod\ j)}, \mathbb{N}^{\geq 17}, \ldots$

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- Is the following set of ordinality $\mathbb{N}$?

$$\left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$$

Yes

No

Unknown to science

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- Is the following set of ordinality $\mathbb{N}$?

$$\left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$$

0, 1, 2, 3, …

$$f(n) = \frac{2^n - 1}{2^n}$$

Yes

No

Unknown to science

Another Ordering

- Consider ordering

$$0 < \frac{1}{2} < \frac{3}{4} < \cdots < 1 < \frac{3}{2} < \frac{7}{4} < \cdots$$

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$$\mathbb{N} + \mathbb{N}$$

How Do $\mathbb{N} + \mathbb{N}$ and $\mathbb{Z}$ Compare?

$\mathbb{N} + \mathbb{N} \preccurlyeq \mathbb{Z}$

$\mathbb{N} + \mathbb{N} \succcurlyeq \mathbb{Z}$

Incomparable

Unknown to science

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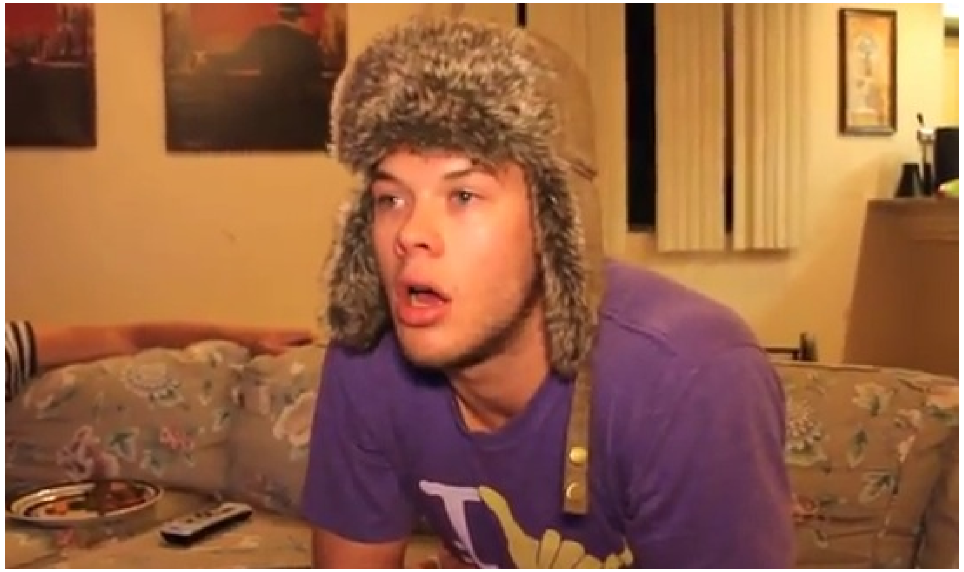

$\mathbb{N} + \mathbb{N} \nprec \mathbb{Z}$

- Recall: $\mathbb{N} + \mathbb{N}$ is:

$$0 < \frac{1}{2} < \frac{3}{4} < \cdots < 1 < \frac{3}{2} < \frac{7}{4} < \cdots$$

- While $\mathbb{Z}$ is:

$$\ldots < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$$

- To say that $\mathbb{N} + \mathbb{N} \prec \mathbb{Z}$would be equivalent to saying that there exists an OPI from $\mathbb{N} + \mathbb{N}$ to $\mathbb{Z}$.

$\mathbb{N} + \mathbb{N} \not\prec \mathbb{Z}$

- $\mathbb{N} + \mathbb{N}$ is:

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$\mathbb{N} + \mathbb{N} \nless \mathbb{Z}$

- $\mathbb{N} + \mathbb{N}$ is:

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$$\ldots < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots < 499 < 500 < \cdots$$

- Suppose we actually do have an OPI $f: \mathbb{N} + \mathbb{N} \to \mathbb{Z}$ such that $f(0) = -3,\ f(1) = 500.$

$\mathbb{N} + \mathbb{N} \nprec \mathbb{Z}$

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- Suppose we actually do have an OPI $f: \mathbb{N} + \mathbb{N} \to \mathbb{Z}$ such that $f(0) = -3,\ f(1) = 500.$
  - Impossible, since there are 504 elements between -3 and 500 in $\mathbb{Z}$ (finite number), while there are **infinite elements** between 0 and 1 in $\mathbb{N} + \mathbb{N}$!

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  - Impossible, since there are 504 elements between -3 and 500 in $\mathbb{Z}$ (finite number), while there are **infinite elements** between 0 and 1 in $\mathbb{N} + \mathbb{N}$!
  - Therefore, no such OPI f can exist.

$\mathbb{Z} \nprec \mathbb{N} + \mathbb{N}$

- It is also the case that $\mathbb{Z}$ cannot be injected (with a preserved ordering) into $\mathbb{N} + \mathbb{N}$
- Recall: $\mathbb{N} + \mathbb{N}$ is:

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- While $\mathbb{Z}$ is:

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$\mathbb{Z} \nprec \mathbb{N} + \mathbb{N}$

- Suppose that we have such an OPI f from $\mathbb{Z}$ to $\mathbb{N} + \mathbb{N}$.

$$0 < \frac{1}{2} < \frac{3}{4} < \cdots < 1 < \frac{3}{2} < \frac{7}{4} < \cdots$$

$$\ldots < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$$

$\mathbb{Z} \nless \mathbb{N} + \mathbb{N}$

- Suppose that we have such an OPI f from $\mathbb{Z}$ to $\mathbb{N} + \mathbb{N}$.

$$0 < \frac{1}{2} < \frac{3}{4} < \cdots < 1 < \frac{3}{2} < \frac{7}{4} < \cdots$$

f

$$\ldots < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$$

- Suppose $f(-3) = 0$. Then, what would $f(-4)$ be?

$\mathbb{Z} \nprec \mathbb{N} + \mathbb{N}$

- Suppose that we have such an OPI f from $\mathbb{Z}$ to $\mathbb{N} + \mathbb{N}$.

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- Suppose $f(-3) = 0$. Then, what would $f(-4)$ be?
  - Since f is **order-preserving,** there **is no such element** in $\omega + \omega$!
  - Therefore, no such OPI f can possibly exist!

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How Do $\mathbb{N} + \mathbb{N}$ and $\mathbb{Q}$ Compare?

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- We leave the proofs of both $\mathbb{N} + \mathbb{N} \preccurlyeq \mathbb{Q}$ and $\mathbb{N} + \mathbb{N} \not\succcurlyeq \mathbb{Q}$ to you.

Take-Home Message

- Orders **can be non-comparable.**
- Cardinalities are **always comparable.**

STOP

RECORDING