START RECORDING

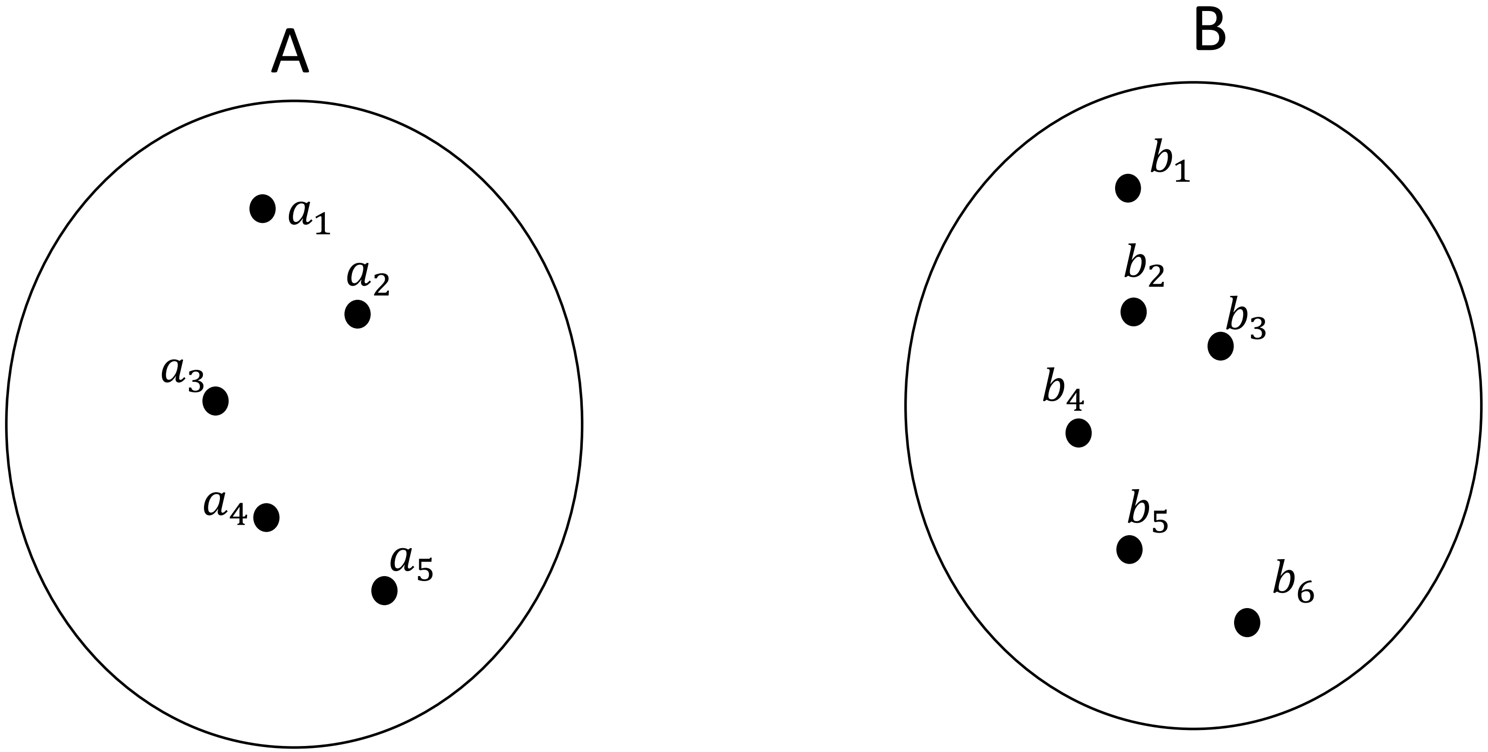
Relations and Functions

CMSC250

Relations

Arrow diagrams

- ***Any*** subset of $A \times B$ is called a ***relation*** from A to B.



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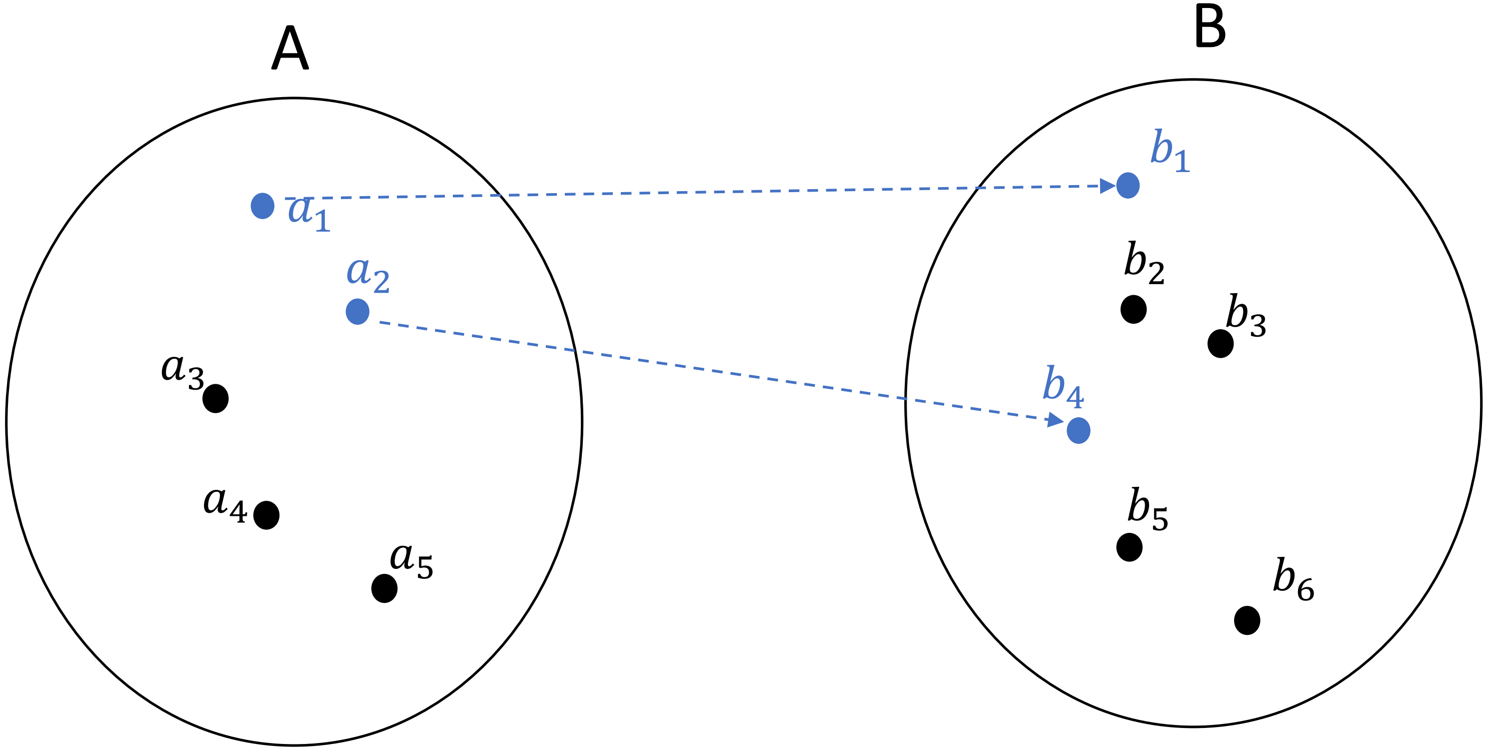

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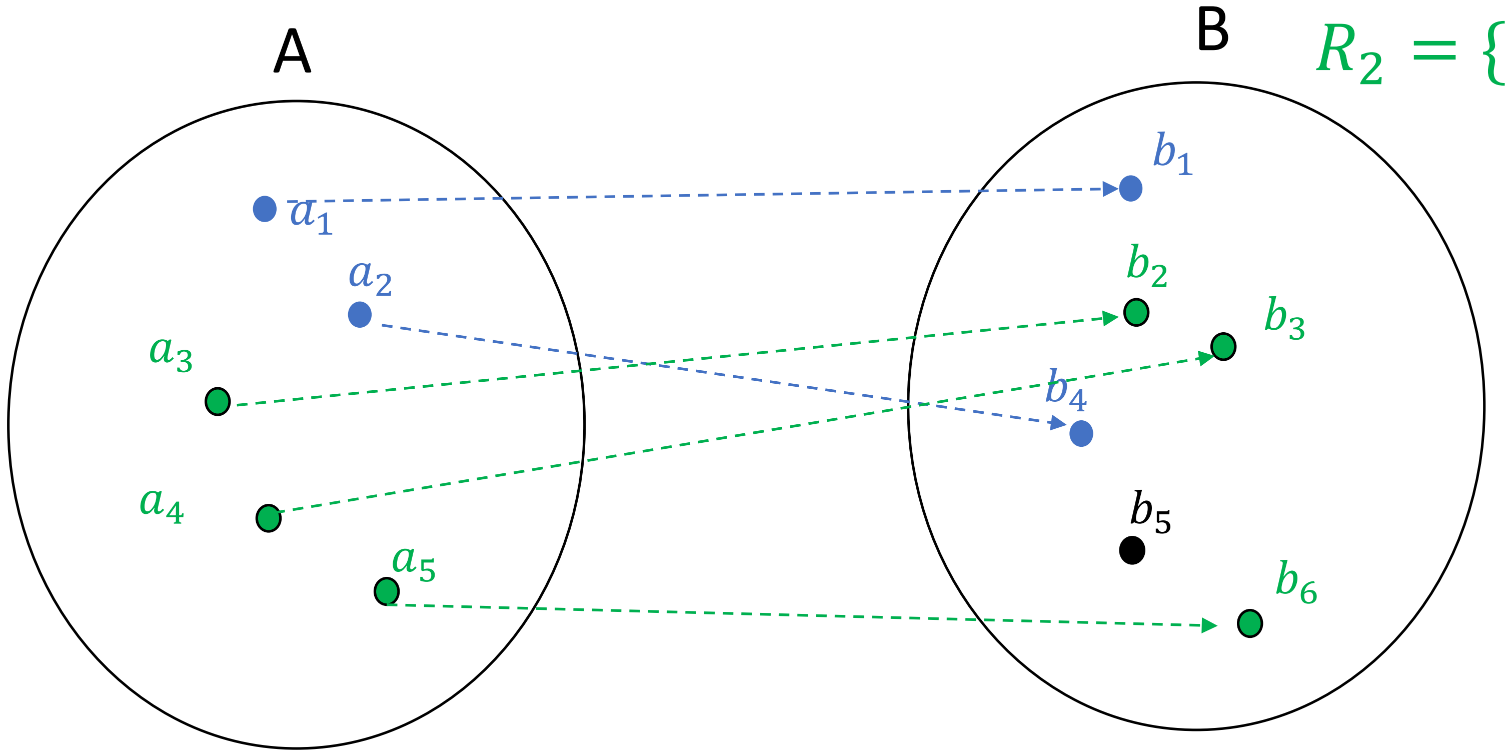
$R_2 = \{(a_3, b_2), (a_4, b_3), (a_5, b_6)\}$



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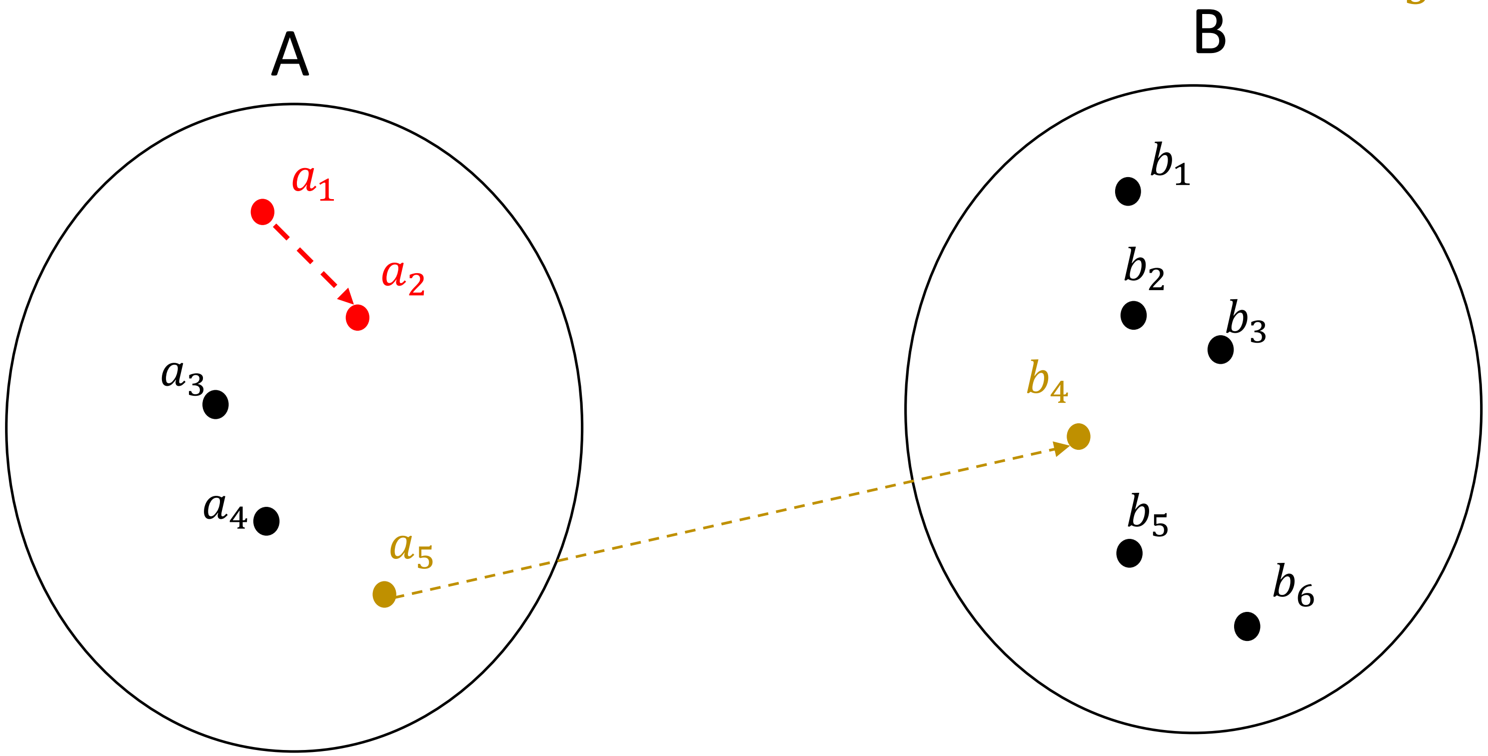
$R_3 = \{(a_1, a_2), (a_5, b_4)\}$



Is **not** a relation, since it contains an element ((a_1, a_2)) which is ***not*** in $A \times B$.

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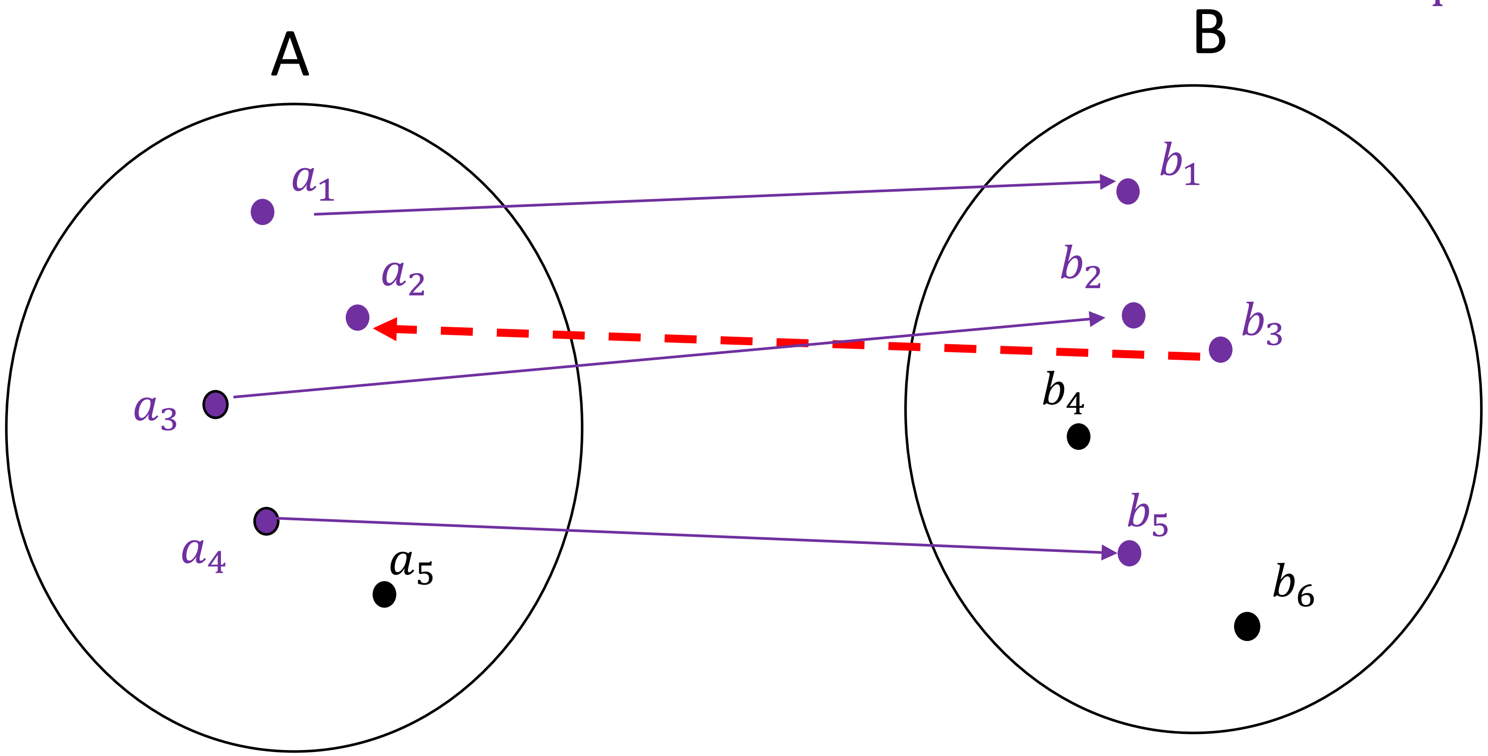

$R_4 = \{(a_1, b_1), (a_3, b_2), (a_4, b_5), (b_3, a_2)\}$

Is ***not*** a relation, since it contains an element ((b_3, a_2)) which is ***not*** in $A \times B$.

Definition

- Let A, B be sets. A relation R from A to B is any subset of $A \times B$.

Examples

- $(<, \mathbb{R} \times \mathbb{R})$
  - $\{\dots, (-1.5, -1.2), (-1.4, -1.2), (\sqrt{2}, \sqrt{3}), (\sqrt{2}, \sqrt{5}), \dots\}$

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  - $\{\dots, (2, 2), (2, 2.1), (\sqrt{2}, \sqrt{3}), (\sqrt{2}, \sqrt{5}), \dots\}$
- $(R, \mathbb{R} \times \mathbb{N})$
  - $\{(r, n) \mid n$ appears in the decimal expansion of $r\}$
  - E.g: $\{\dots, (\pi, 1),\ (e, 7),\ ({}^1/_3, 3), \dots\}$
  - We would formally say that all of the above are elements of the relation R

Reflexivity

- A relation $\mathrm{X} \subseteq A \times A$ is **reflexive** if

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- Examples:
  - $(\leq, \mathbb{N} \times \mathbb{N})$ is reflexive, since $(\forall n \in \mathbb{N})[n \leq n]$
  - $(<, \mathbb{N} \times \mathbb{N})$ is **not** reflexive, since $\sim (\forall n \in \mathbb{N})[n < n]$ (in fact, there is no such n)
  - $(R, \mathbb{N} \times \mathbb{N})$ defined as $\{(x, y) \mid x + y \geq 100\}$ is **not** reflexive (e.g $10 \in \mathbb{N}$, but $(10, 10) \notin R$)

Symmetry

- A relation $\mathrm{X} \subseteq A \times A$ is **symmetric** if

$$(\forall a_1, a_2 \in A)\left[\left((a_1, a_2) \in X\right) \Rightarrow ((a_2, a_1) \in X)\right]$$

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- Examples:
  - $(\leq, \mathbb{N} \times \mathbb{N})$ is **not** symmetric since $4 \leq 5$ but $\sim (5 \leq 4)$
  - $(<, \mathbb{N} \times \mathbb{N})$ is **not** symmetric (see above)
  - $(R, \mathbb{N} \times \mathbb{N})$ defined as $\{(x, y) \mid x + y \geq 100\}$ **is symmetric** since

$$(x + y \geq 100) \Rightarrow (y + x \geq 100)$$

Transitivity

- A relation $\mathrm{X} \subseteq A \times A$ is **transitive** if

$$(\forall a_1, a_2, a_3 \in A)\left[\left((a_1, a_2) \in X\right) \wedge ((a_2, a_3) \in X) \Rightarrow (a_1, a_3) \in X)\right]$$

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- $(\leq, \mathbb{N} \times \mathbb{N})$ **is transitive** since $\big((x \leq y) \wedge (y \leq z)\big) \Rightarrow (x \leq z)$
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Transitivity

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$$(\forall a_1, a_2, a_3 \in A)\Big[\big((a_1, a_2) \in X\big) \land ((a_2, a_3) \in X) \Rightarrow (a_1, a_3) \in X)\ \Big]$$

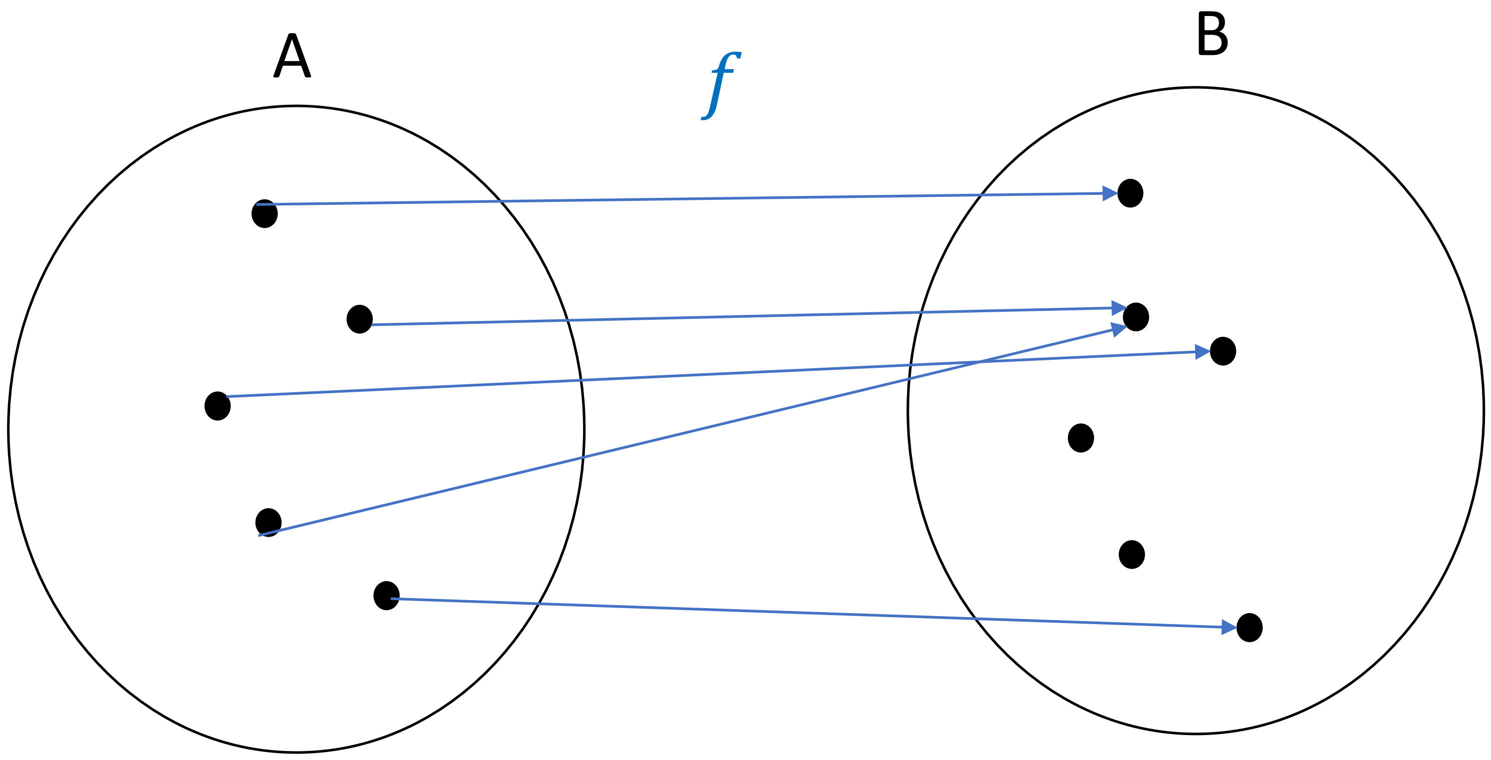
- $(\leq, \mathbb{N} \times \mathbb{N})$ **is transitive** since $\big((x \leq y) \land (y \leq z)\big) \Rightarrow (x \leq z)$
- $(<, \mathbb{N} \times \mathbb{N})$ **is transitive** (see above)
- $(R, \mathbb{N} \times \mathbb{N})$ defined as $\{(x, y) \mid x + y \geq 100\}$ **is not transitive** since (counter-example):

$$((1, 100) \in R) \land ((100, 5) \in R), \text{ but } (1, 5) \notin R$$

Functions

Functions

- Most basic representation: **Arrow Diagrams**



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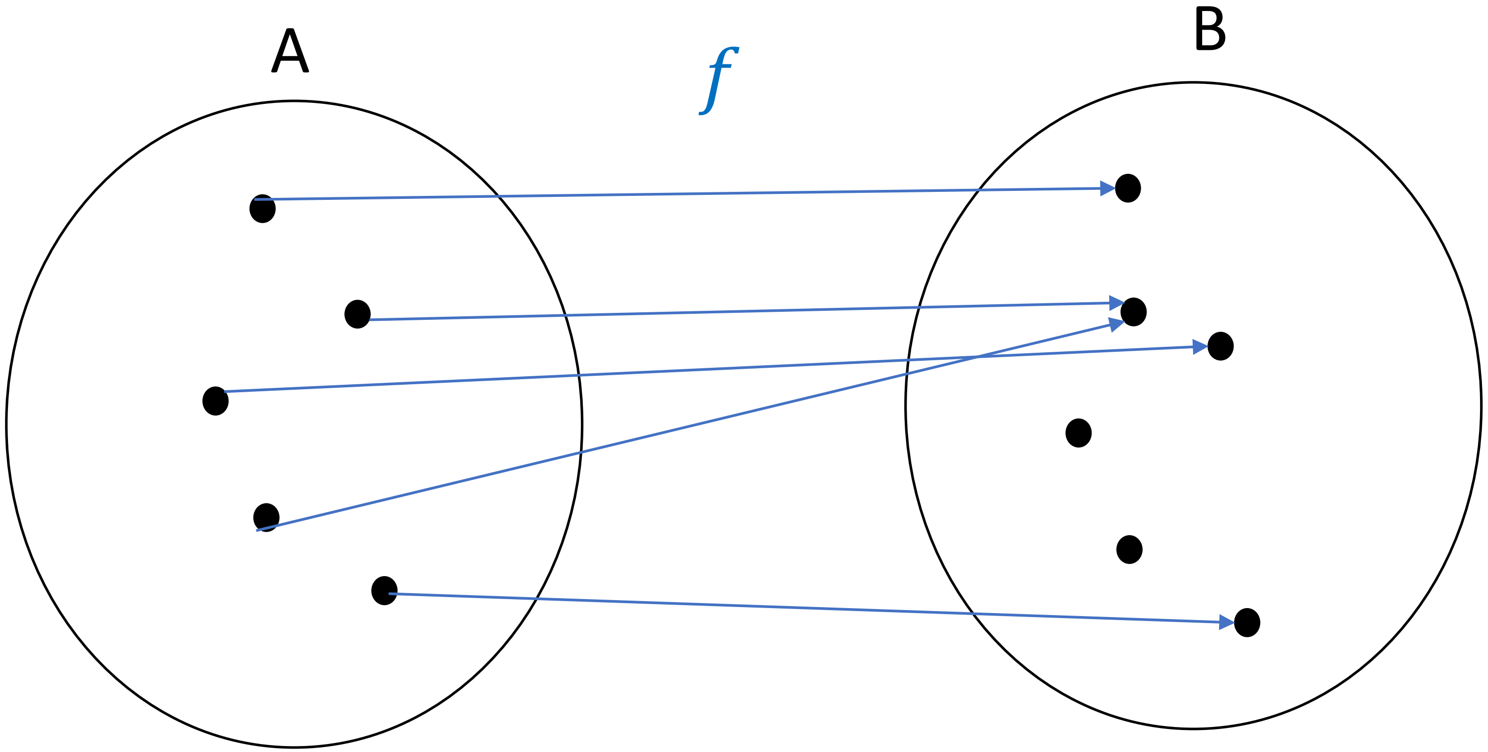

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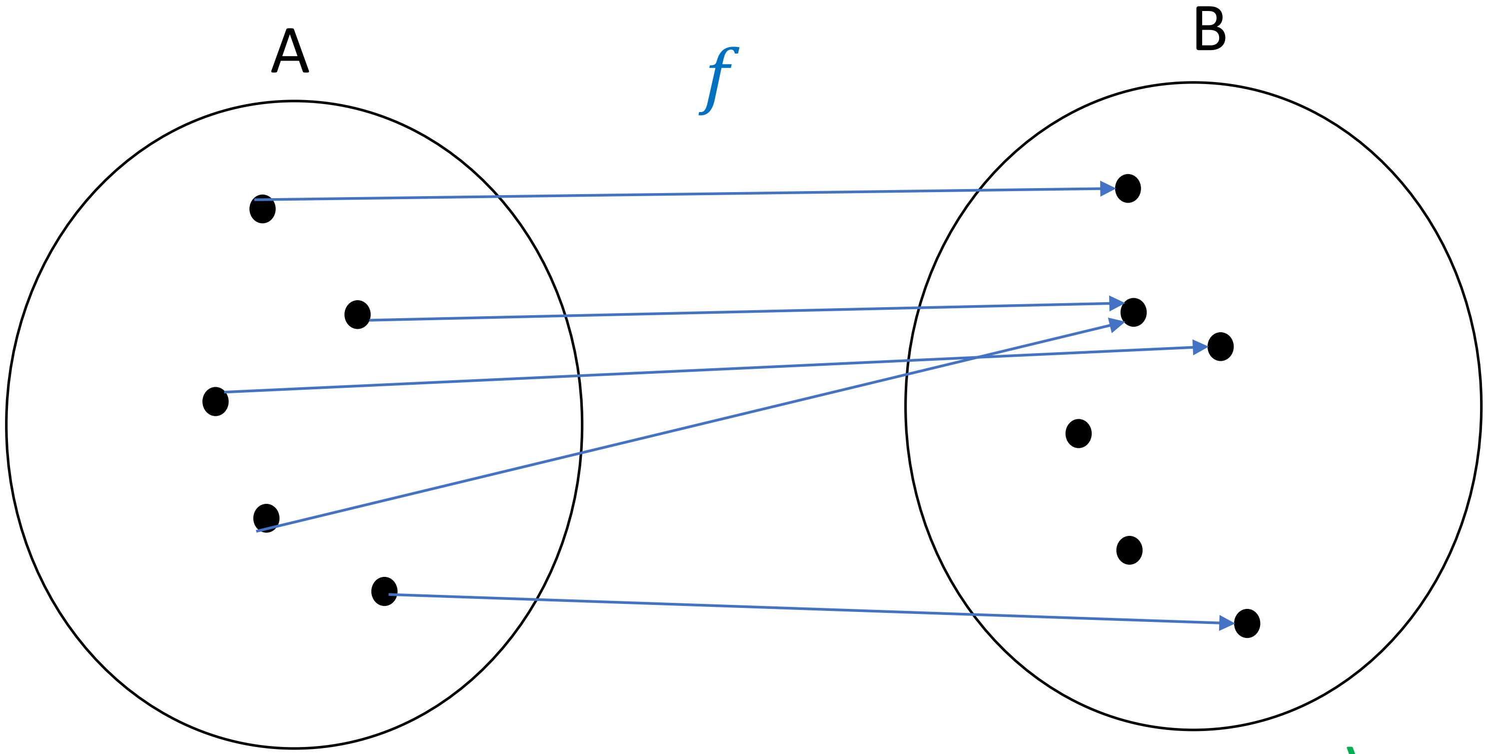

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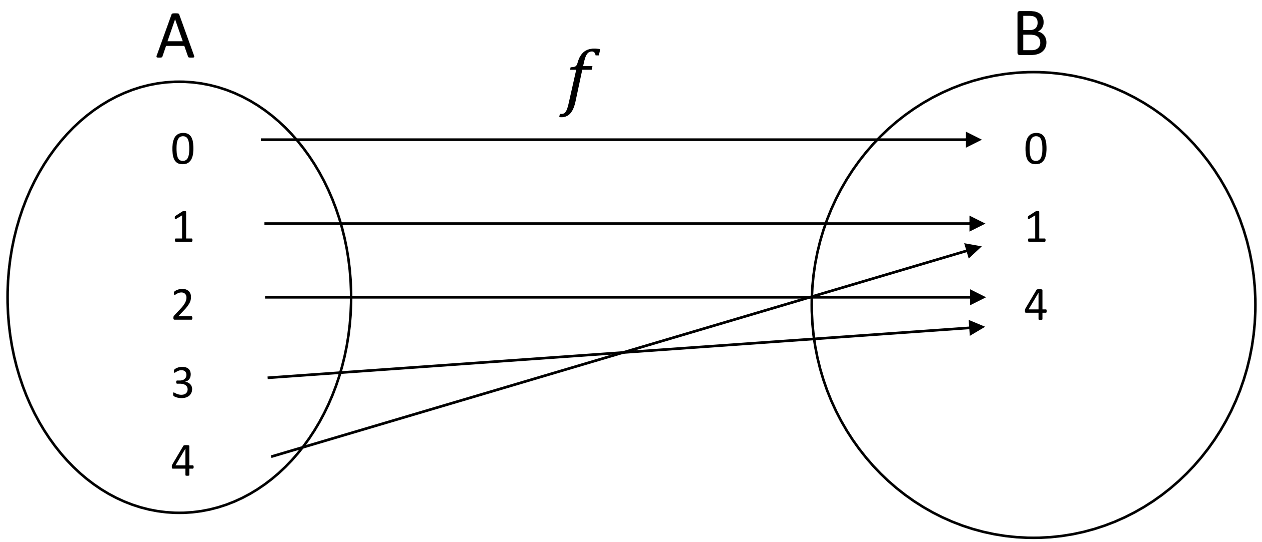
\mapsto in LaTeX

Example 1

- Is this a function?

Yes

No



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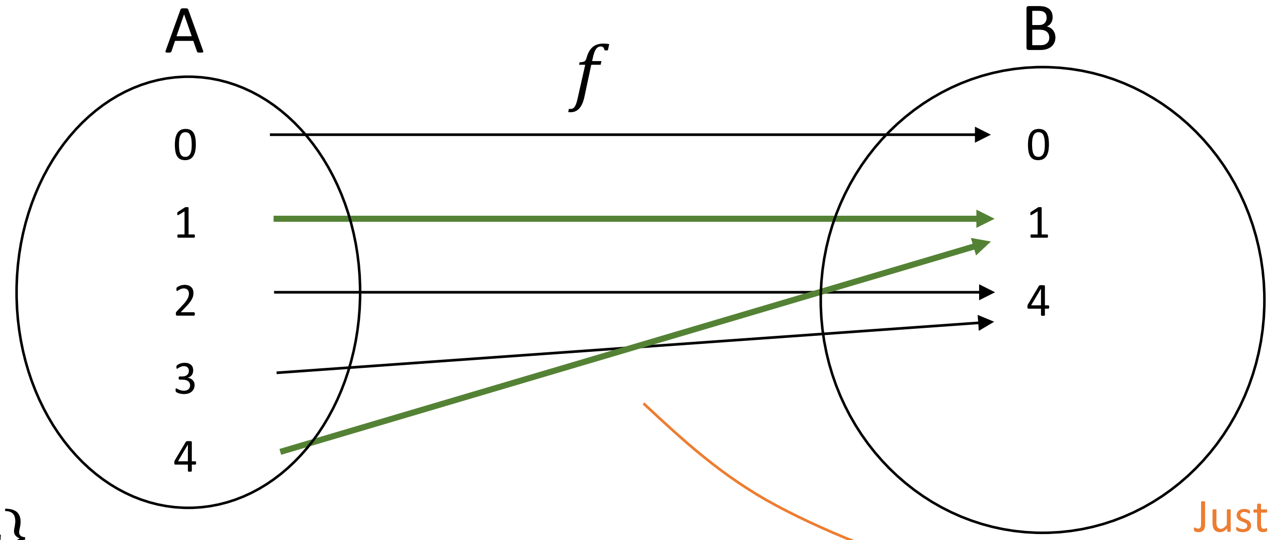


- Domain: $\{0, 1, 2, 3, 4\}$
- Co-domain: $\{0, 1, 4\}$
- Formula (that we came up with): $f(x) = x^2 \textbf{ mod } 5$

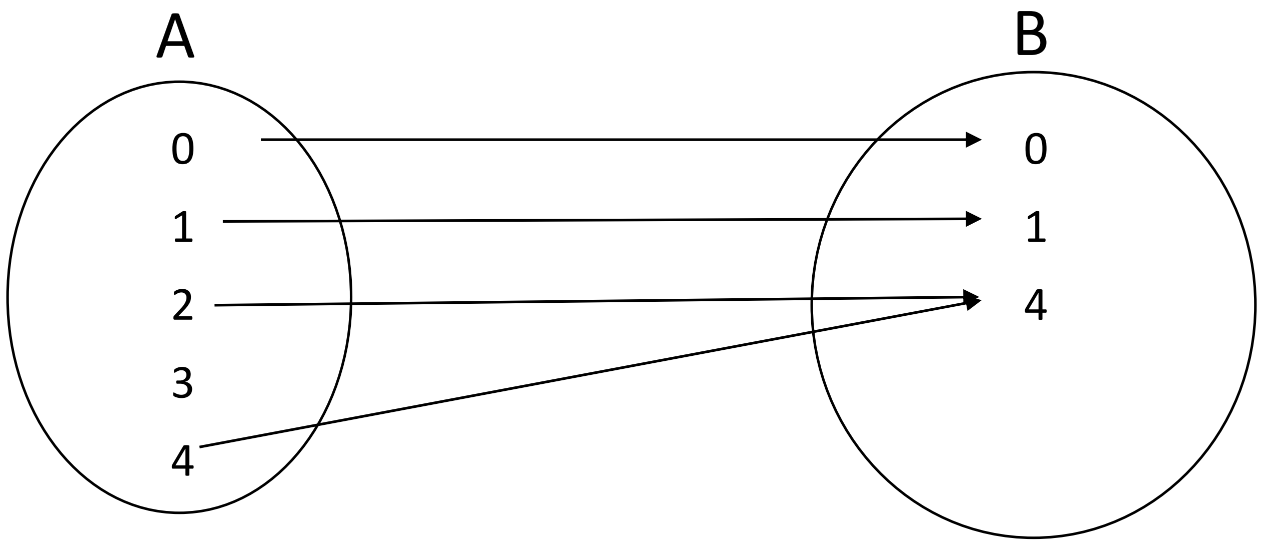
Just because two ‘x’s map to the same ‘y’ doesn’t make this a non-function… it just makes it a **non-injective** (not “1-1”) function

Example 2

- Is this a function?

Yes

No



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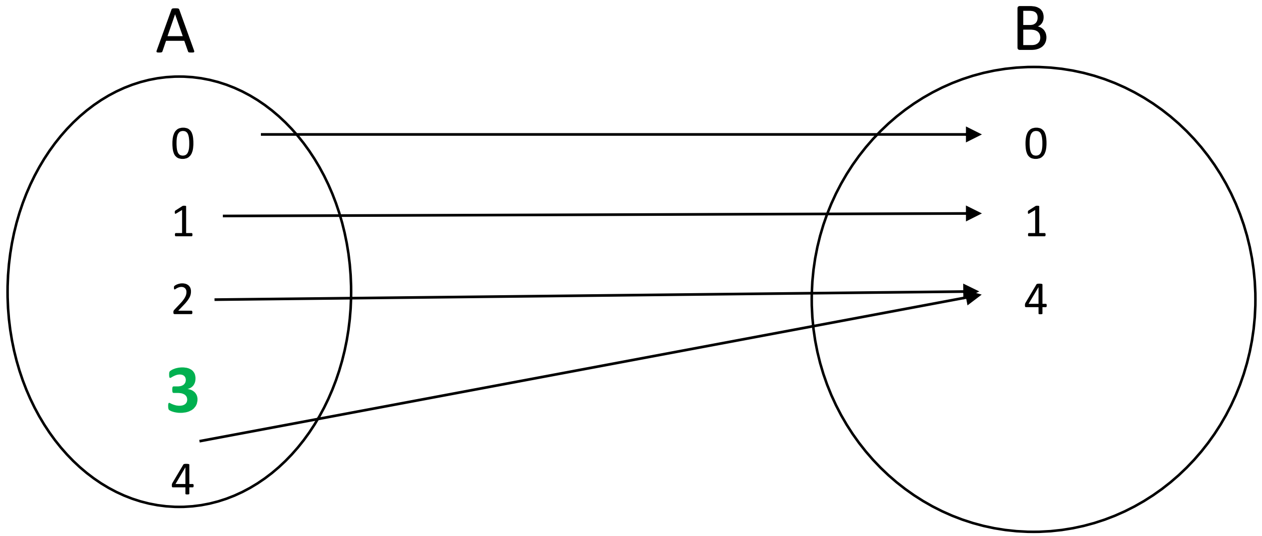

- **Every element of the domain should map to some co-domain element!**

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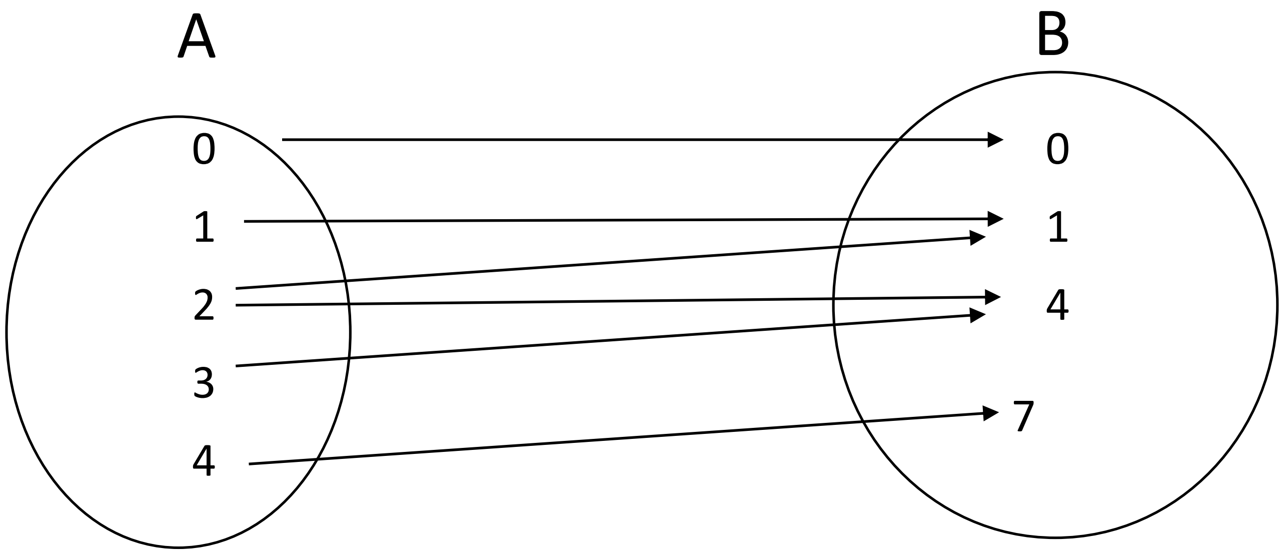

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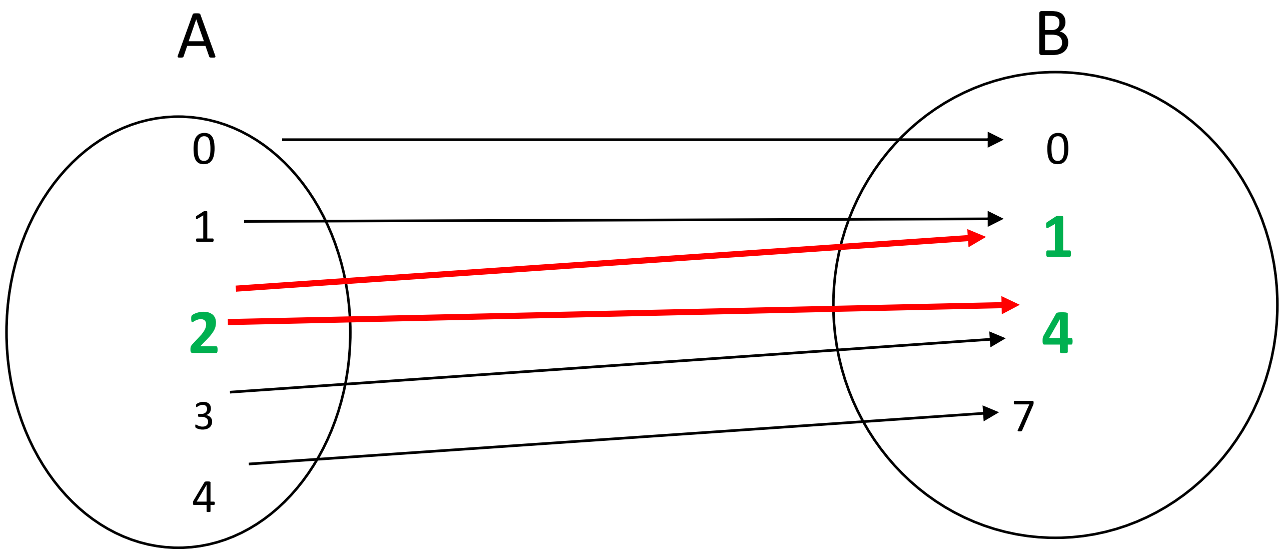

Fails the “vertical line” test (2 different `y’s mapped to by the same `x’)

Example 3

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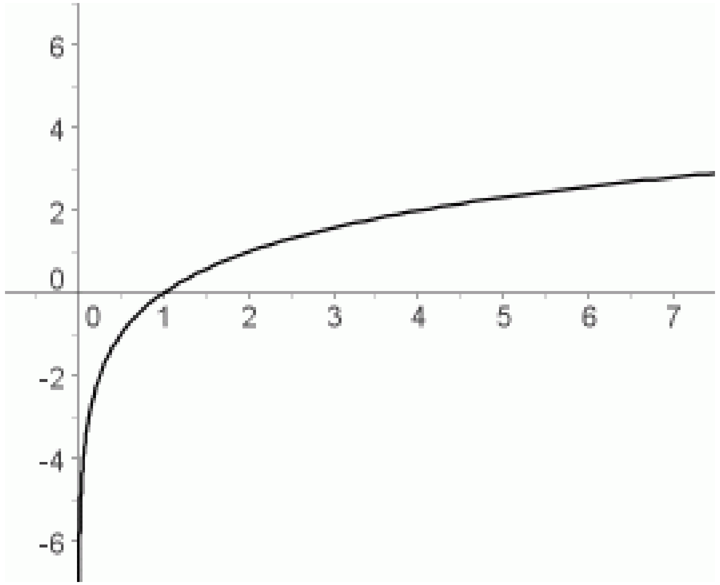
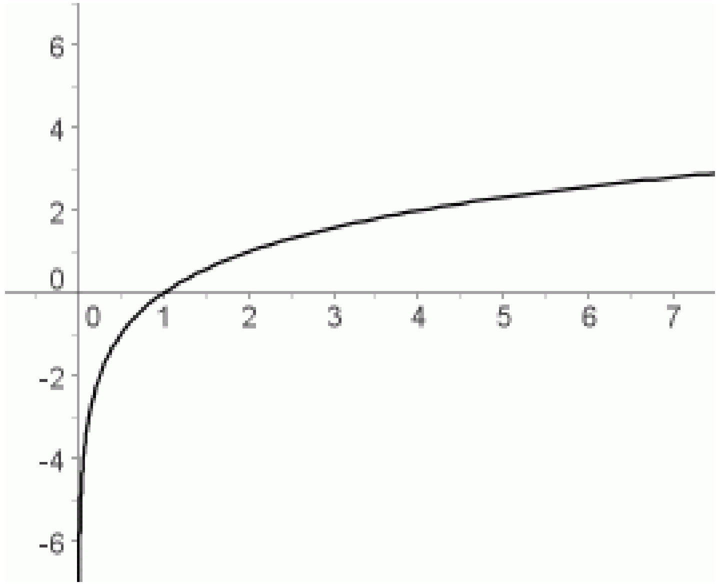


Example 4

Yes

No

(As long as the domain is $\mathbb{R}^{>0}$!!)



Log function

Example 5

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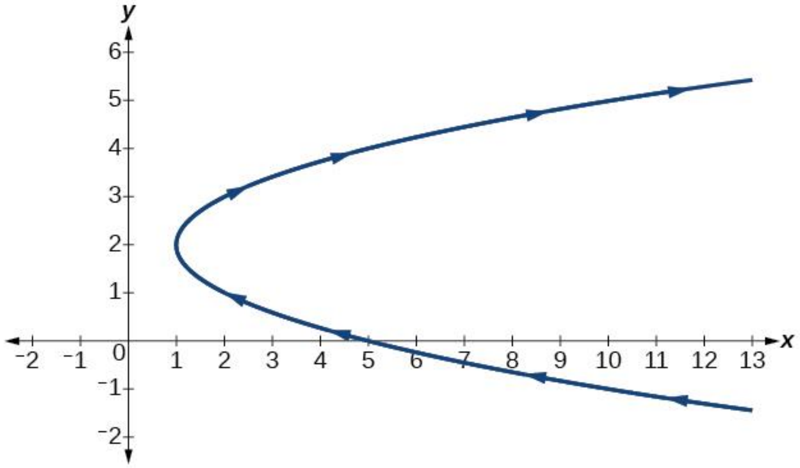

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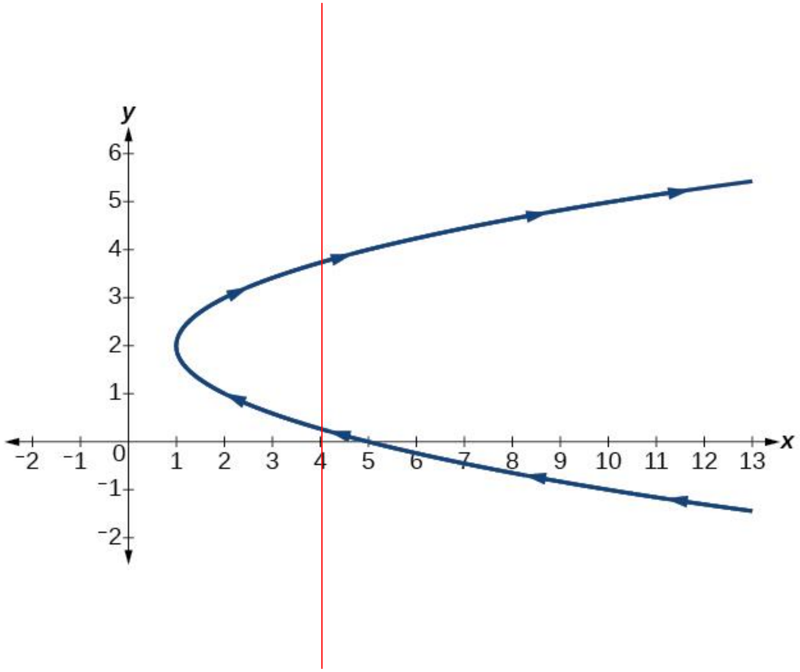

Fails the "vertical line" test

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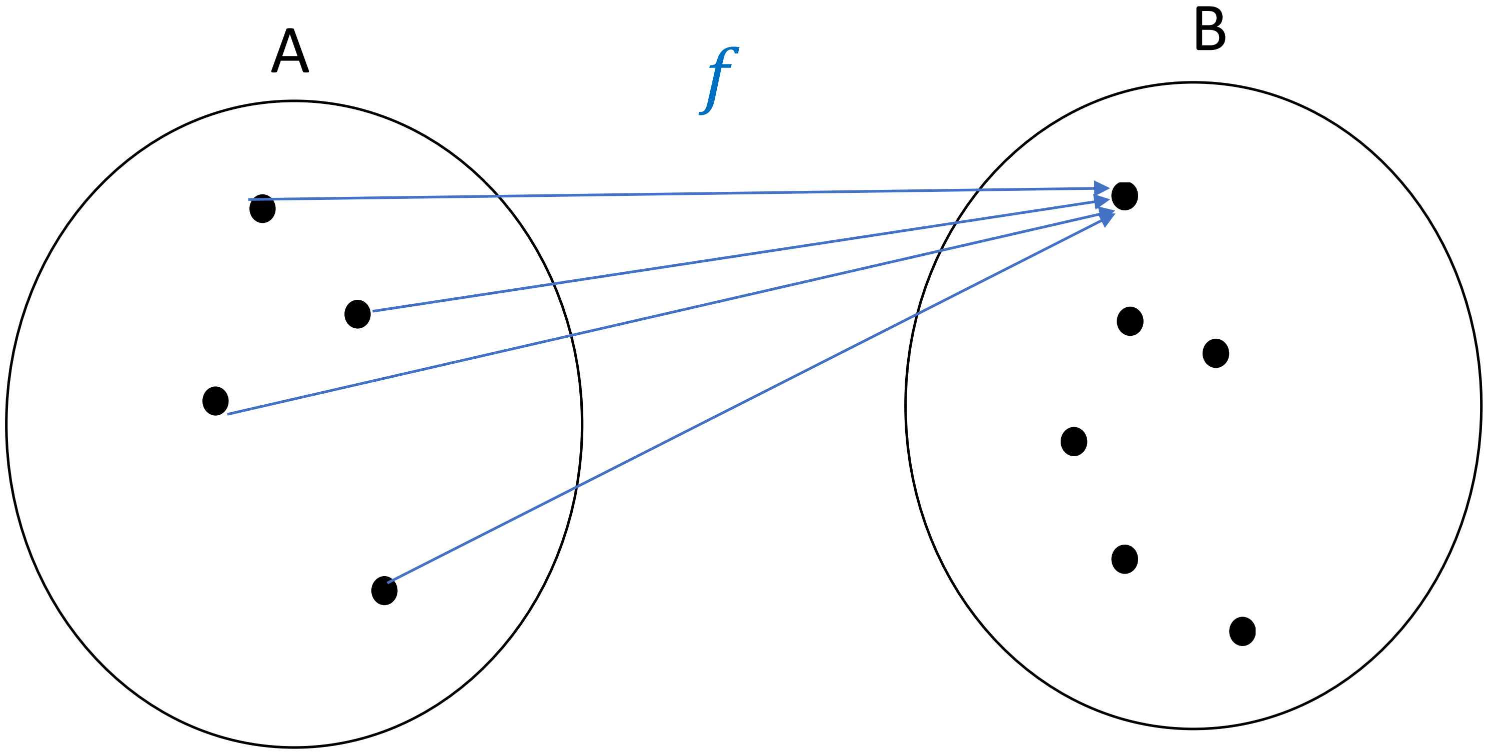

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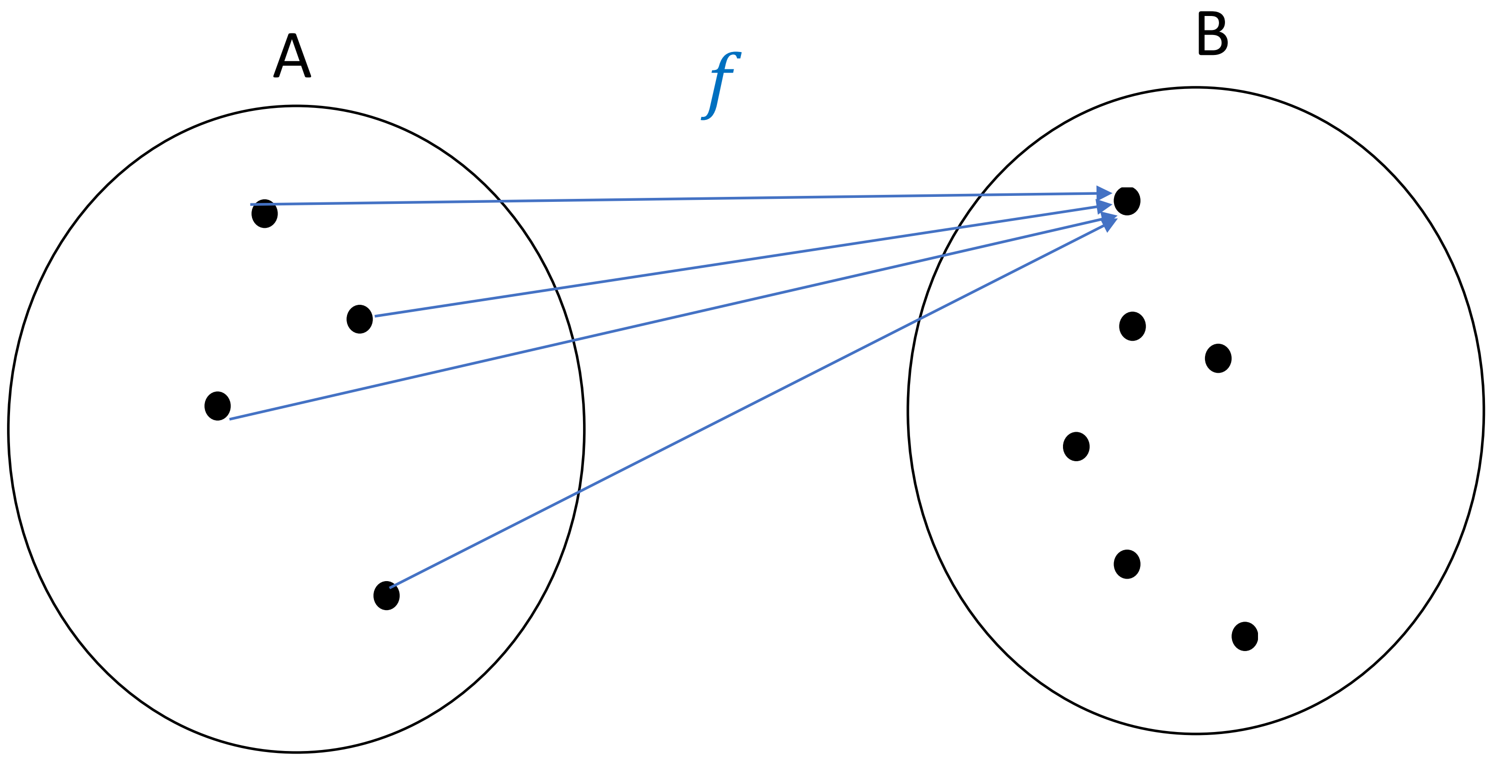

It just happens to not be "1-1"

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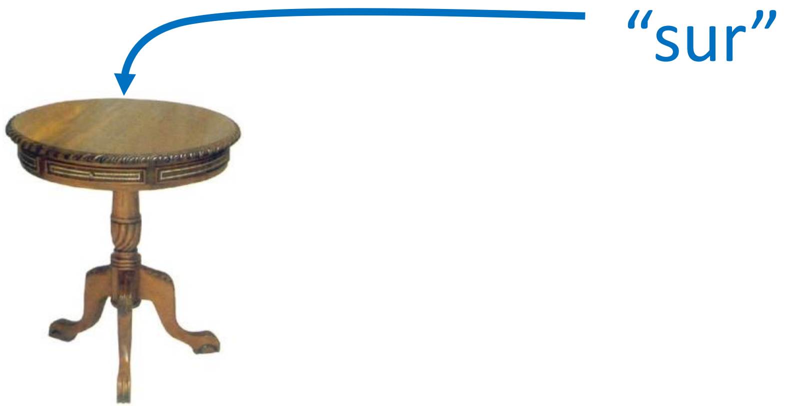

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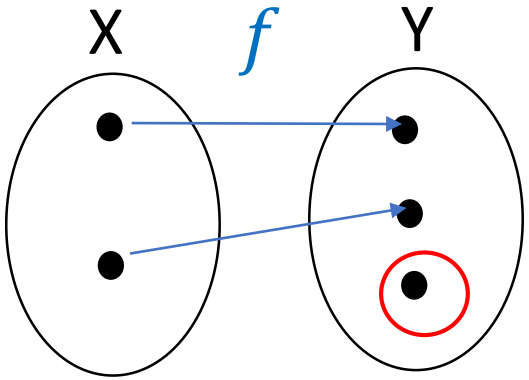
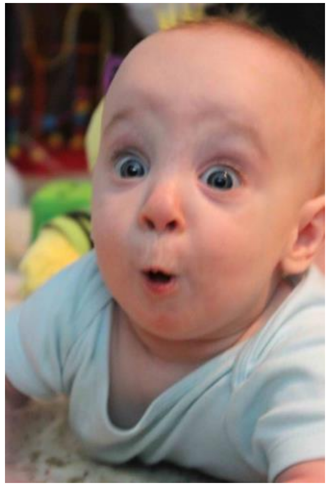
  - ***Why at most one and not exactly one?***
  - ***Because 1-1 but not onto functions are possible!***





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- Can this function **ever** be injective?

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- Can this function **ever** be injective?
  - Yes. Pick $D = \mathbb{N}, C = SQUARES$

Making functions onto or 1-1

- To make a function **onto**, we need to make the **co-domain smaller.**

- To make a function **1-1** , we need to make the **domain smaller.**

Bijective functions

- A function $f: X \mapsto Y$ is called **bijective (or a bijection, or a <u>1-1 correspondence</u>)** iff it is both surjective and injective.
  - We will try to avoid using the term **"1-1 correspondence"** (some books uses it) since it can confuse us with the notion of an injective (or 1-1) **function**.

Quiz on bijections

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  (In all examples, $C = \mathbb{R}$*)*

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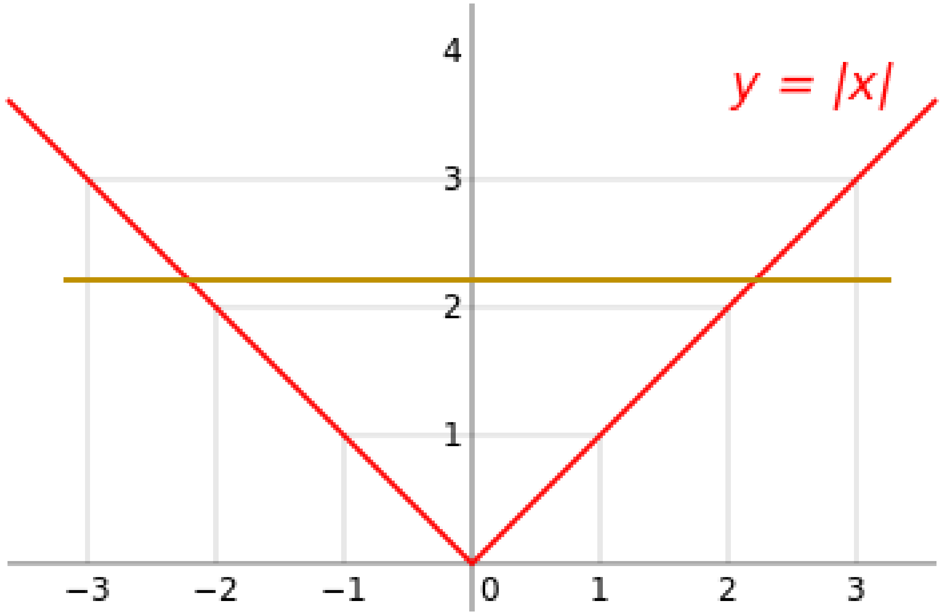

Non-injective!

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Quiz on bijections

Yes | No

- Are the following functions **bijections**?

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!!!!

For a = 0, the graph of the function fails the “horizontal line test”!

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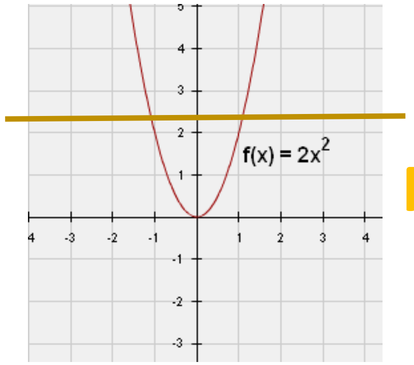

Non-injective!

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Non-surjective! Set $h(n) = y$ and solve for n:

$$4n - 1 = y \Rightarrow n = \frac{y+1}{4}$$

There are infinitely many choices of y for which $n \notin \mathbb{Z}$!

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4. $h(n) = 4n - 1, n \in \mathbb{Z}$ *No*
5. $h(x) = 4x - 1, x \in \mathbb{R}$ *Yes*

Surjective and injective! Surjective, since, if we set $h(n) = y$ and solve for n:

$$4n - 1 = y \Rightarrow n = \frac{y+1}{4}$$

For every real y, there's always a **real** solution n. Injective, since it's of the form of (2) with $a \neq 0$.

Functions in history

- Pre-modern views:

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Gottfried Wilhelm Leibniz

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Leonard Euler

Functions in history

- Pre-modern view:
  1. Leibniz: Limited to algebraic or transcendental functions (like sin, cos, etc)
  2. Euler: An expression or formula.
  3. Also Euler: If x changes "a little", so should $f(x)$.
- In general, people considered "functions" to only be differentiable maps from $\mathbb{R}$ to $\mathbb{R}$.
- The reason for the restriction because of the tight coupling of math and physics at that time: a "function" was something that could come up in nature.

Functions in history

- The view began to change around the era of Fourier.
- While studying heat, Fourier found out that the following function is discontinuous:

$$f(x, y, z, t) = \text{temperature at loc } (x, y, z) \text{ at time } t.$$

- Lighting a match at $(0, 0, 0)$ introduces a discontinuity in the function.

Joseph Fourier

Functions in history

- For a while people tried to extend the notion of function.
- Charles Hermite: “"I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives.”

Charles Hermite, pictured here turning away with terror and horror from the lamentable scourge of continuous functions with no derivatives.

Modern View

- Owed primarily to Dirichlet and Lobachevsky.
- According to this view, ***any correspondence*** is a function.

Emanuel Dirichlet

Nicolai Lobachevsky

STOP RECORDING