# The Roots Hierarchy <br> Exposition by William Gasarch and Erik Metz 

## 1 Introduction

The main proof in this note is from Problems from the Book by Dospinescu and Andreescu.
We want to classify real numbers in terms of their complexity.

Def 1.1 Let $d \in \mathrm{~N}$.

1. $\mathrm{Z}_{d}[x]$ is the set of polynomials of degree $d$ over $\mathbf{Z}$ (the integers).
2. roots $_{d}$ is the set of roots of polynomials in $Z_{d}[x]$. Note that roots $_{1}=\mathbf{Q}$.

Clearly $\operatorname{roots}_{1} \subseteq \operatorname{roots}_{2} \subseteq \operatorname{roots}_{3} \subseteq \cdots$
We want to show that $\operatorname{roots}_{1} \subset \operatorname{roots}_{2} \subset \operatorname{roots}_{3} \subset \cdots$

## 2 The Hierarchy is Proper

We show that roots $3_{3} \subset$ roots $_{4}$. All of the ideas to show $\operatorname{roots}_{d-1} \subset \operatorname{roots}_{d}$ are contained in the proof. The main method for the proof is taken from chapter 9 of Problems from the Book by Titu Andreescu and Gabriel Dospinescu.

Theorem 2.1 roots $_{3} \subset$ roots $_{4}$.

Proof: Clearly roots $3 \subseteq$ roots $_{4}$. We show that $2^{1 / 4} \in$ roots $_{4}-$ roots $_{3}$ which implies

$$
\operatorname{roots}_{3} \subset \operatorname{roots}_{4}
$$

Clearly $2^{1 / 4}$ is a root of $x^{4}-2=0$ and hence $2^{1 / 4} \in$ roots $_{4}$. We show that $2^{1 / 4} \notin \operatorname{roots}_{3}$ Assume, by way of contradiction, that there exists $a_{0}, a_{1}, a_{2}, a_{3} \in \mathbf{Z}$ such that

$$
a_{3}\left(2^{1 / 4}\right)^{3}+a_{2}\left(2^{1 / 4}\right)^{2}+a_{1}\left(2^{1 / 4}\right)+a_{0}=0
$$

which is

$$
a_{3} \times 2^{3 / 4}+a_{2} \times 2^{1 / 2}+a_{1} \times 2^{1 / 4}+a_{0} \times 1=0
$$

We assume the following about $\left(a_{3}, a_{2}, a_{1}, a_{0}\right)$ : They are not all even. If they are then divide each one by 2 to get a smaller poly over $Z$ and use that.

Multiply this equation by $1,2^{1 / 4}, 2^{1 / 2}, 2^{3 / 4}$ to get

$$
\begin{aligned}
& a_{3} \times 2^{3 / 4}+a_{2} \times 2^{1 / 2}+a_{1} \times 2^{1 / 4}+a_{0} \times 1=0 \\
& a_{2} \times 2^{3 / 4}+a_{1} \times 2^{1 / 2}+a_{0} \times 2^{1 / 4}+2 a_{3} \times 1=0 \\
& a_{1} \times 2^{3 / 4}+a_{0} \times 2^{1 / 2}+2 a_{3} \times 2^{1 / 4}+2 a_{2} \times 1=0 \\
& a_{0} \times 2^{3 / 4}+2 a_{3} \times 2^{1 / 2}+2 a_{2} \times 2^{1 / 4}+2 a_{1} \times 1=0
\end{aligned}
$$

We rewrite this as a matrix times a vector being the zero vector:

$$
\left(\begin{array}{cccc}
a_{3} & a_{2} & a_{1} & a_{0} \\
a_{2} & a_{1} & a_{0} & 2 a_{3} \\
a_{1} & a_{0} & 2 a_{3} & 2 a_{2} \\
a_{0} & 2 a_{3} & 2 a_{2} & 2 a_{1}
\end{array}\right)\left(\begin{array}{c}
2^{3 / 4} \\
2^{1 / 2} \\
2^{1 / 4} \\
1
\end{array}\right)=\left(\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right)
$$

Let

$$
A=\left(\begin{array}{cccc}
a_{3} & a_{2} & a_{1} & a_{0} \\
a_{2} & a_{1} & a_{0} & 2 a_{3} \\
a_{1} & a_{0} & 2 a_{3} & 2 a_{2} \\
a_{0} & 2 a_{3} & 2 a_{2} & 2 a_{1}
\end{array}\right)
$$

The matrix above can be multiplied by a non-zero vector and get zero. Hence the matrix has det 0 . Hence the det is 0 MOD 2 .

$$
A \quad(\bmod 2)=\left(\begin{array}{cccc}
a_{3} & a_{2} & a_{1} & a_{0} \\
a_{2} & a_{1} & a_{0} & 0 \\
a_{1} & a_{0} & 0 & 0 \\
a_{0} & 0 & 0 & 0
\end{array}\right)
$$

By the column expansion definition of det, applied to the last row the $\operatorname{det}(\bmod 2)$ is $a_{0}^{4}$, Hence $a_{0}^{4} \equiv 0(\bmod 2)$, so $a_{0} \equiv 0(\bmod 2)$. We rewrite $A$ :

$$
A=\left(\begin{array}{cccc}
a_{3} & a_{2} & a_{1} & 2 b_{0} \\
a_{2} & a_{1} & 2 b_{0} & 2 a_{3} \\
a_{1} & 2 b_{0} & 2 a_{3} & 2 a_{2} \\
2 b_{0} & 2 a_{3} & 2 a_{2} & 2 a_{1}
\end{array}\right)
$$

Since this matrix had det 0 , so does the matrix when I divide the last column by 2 . Hence this matrix has det 0 :

$$
B=\left(\begin{array}{cccc}
a_{3} & a_{2} & a_{1} & b_{0} \\
a_{2} & a_{1} & 2 b_{0} & a_{3} \\
a_{1} & 2 b_{0} & 2 a_{3} & a_{2} \\
2 b_{0} & 2 a_{3} & 2 a_{2} & a_{1}
\end{array}\right)
$$

Take this matrix mod 2 to get:

$$
B \quad(\bmod 2)=\left(\begin{array}{cccc}
a_{3} & a_{2} & a_{1} & b_{0} \\
a_{2} & a_{1} & 0 & a_{3} \\
a_{1} & 0 & 0 & a_{2} \\
0 & 0 & 0 & a_{1}
\end{array}\right)
$$

If you expand the det of $B(\bmod 2)$ on the last row you get $a_{1}^{4}$. Hence $a_{1}^{4} \equiv 0(\bmod 2)$, so $a_{1} \equiv 0(\bmod 2)$. Hence $a_{1}=2 b_{1}$. We rewrite $B$ :

$$
B=\left(\begin{array}{cccc}
a_{3} & a_{2} & 2 b_{1} & b_{0} \\
a_{2} & 2 b_{1} & 2 b_{0} & a_{3} \\
2 b_{1} & 2 b_{0} & 2 a_{3} & a_{2} \\
2 b_{0} & 2 a_{3} & 2 a_{2} & 2 b_{1}
\end{array}\right)
$$

We divide the third column by 2 :

$$
C=\left(\begin{array}{cccc}
a_{3} & a_{2} & b_{1} & b_{0} \\
a_{2} & 2 b_{1} & b_{0} & a_{3} \\
2 b_{1} & 2 b_{0} & a_{3} & a_{2} \\
2 b_{0} & 2 a_{3} & a_{2} & 2 b_{1}
\end{array}\right)
$$

Hence

$$
C \quad(\bmod 2)=\left(\begin{array}{cccc}
a_{3} & a_{2} & b_{1} & b_{0} \\
a_{2} & 0 & b_{0} & a_{3} \\
0 & 0 & a_{3} & a_{2} \\
0 & 0 & a_{2} & 0
\end{array}\right)
$$

By the column expansion definition of det, applied to the last row, $a_{2}$ is even. Let
$a_{2}=2 b_{2}$. If a matrix has det 0 and you divide a column by (say) 2 then the matrix still has det 0 . Divide the second column by 2 , and replace all $a_{2}$ by $2 b_{2}$, to get:

$$
\begin{gathered}
D=\left(\begin{array}{cccc}
a_{3} & b_{2} & b_{1} & b_{0} \\
2 b_{2} & b_{1} & b_{0} & a_{3} \\
2 b_{1} & b_{0} & a_{3} & 2 b_{2} \\
2 b_{0} & a_{3} & 2 b_{2} & 2 b_{1}
\end{array}\right) \\
D \quad(\bmod 2)=\left(\begin{array}{cccc}
a_{3} & b_{2} & b_{1} & b_{0} \\
0 & b_{1} & b_{0} & a_{3} \\
0 & b_{0} & a_{3} & 0 \\
0 & a_{3} & 0 & 0
\end{array}\right)
\end{gathered}
$$

By the column expansion definition of det, applied to the last column, $a_{3}$ is even.
We now have that $a_{3}, a_{2}, a_{1}, a_{0}$ are all even. This contradicts are assumption on $\left(a_{3}, a_{2}, a_{1}, a_{0}\right)$.

