START RECORDING

Strong Induction

CMSC 250

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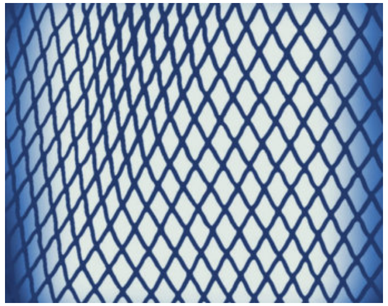
- Visualization:

Weak Induction

Strong Induction



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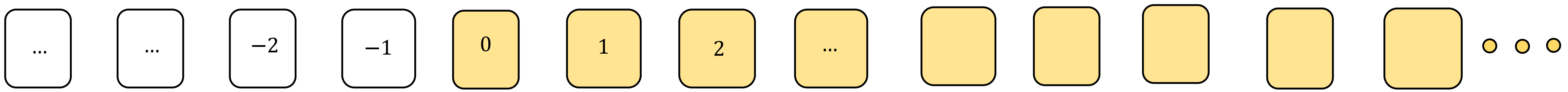

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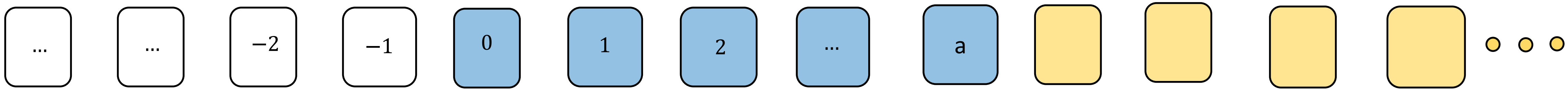

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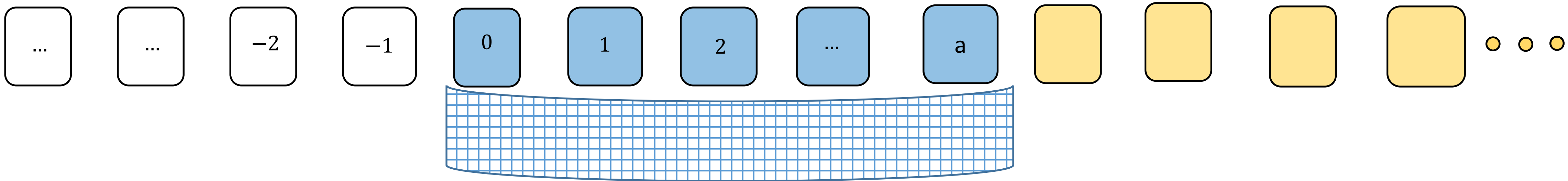

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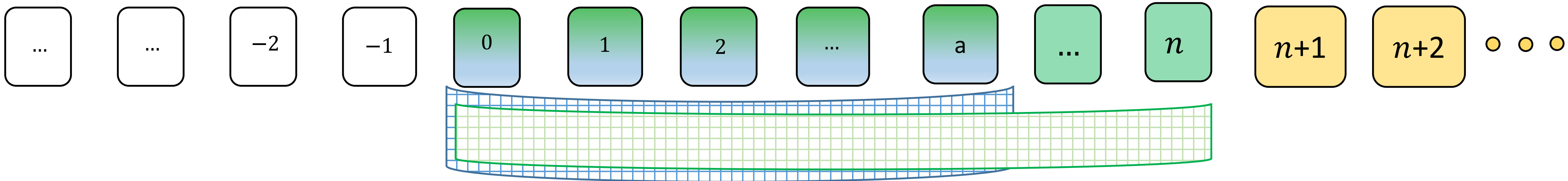

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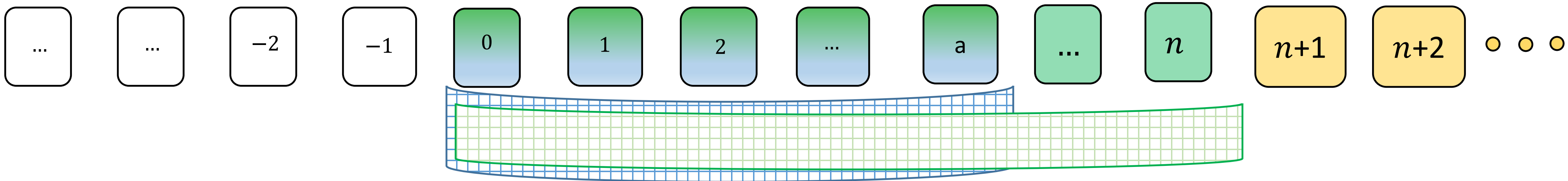

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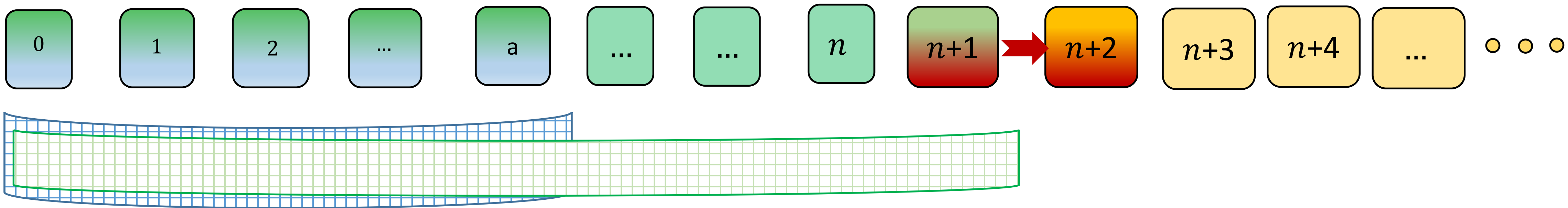

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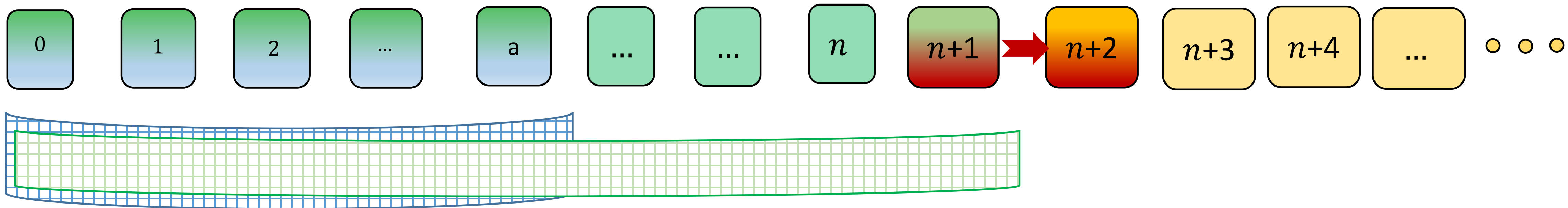

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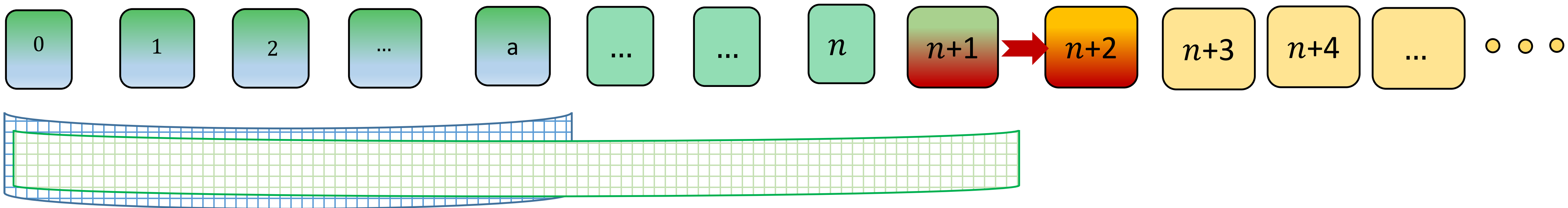
  - Then P(a+1) is true. We can continue with P(a+2), P(a+3), ….



How We'll Make it Work

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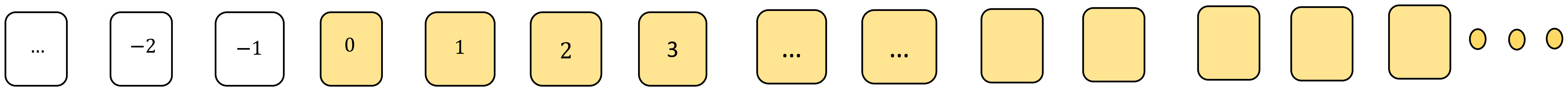

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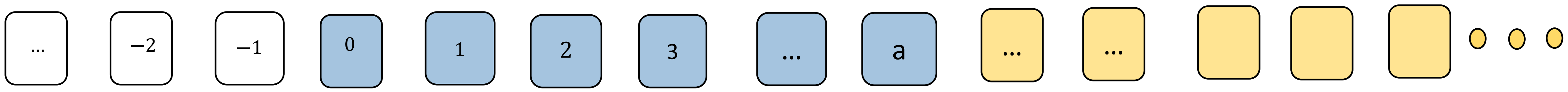

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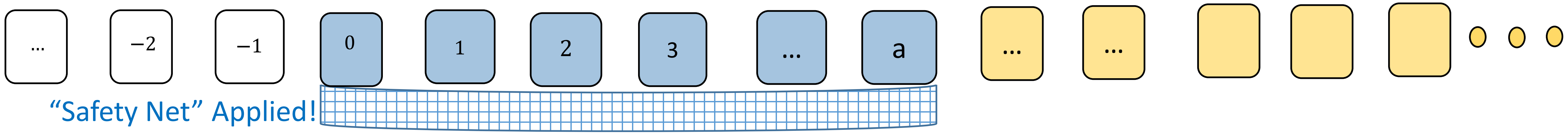

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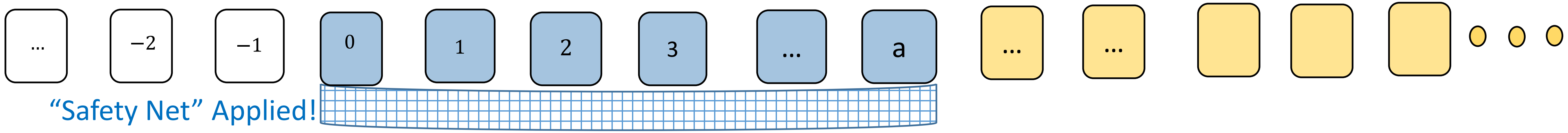

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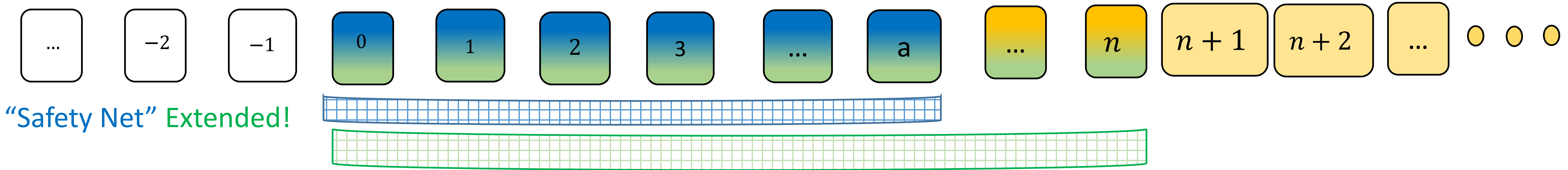

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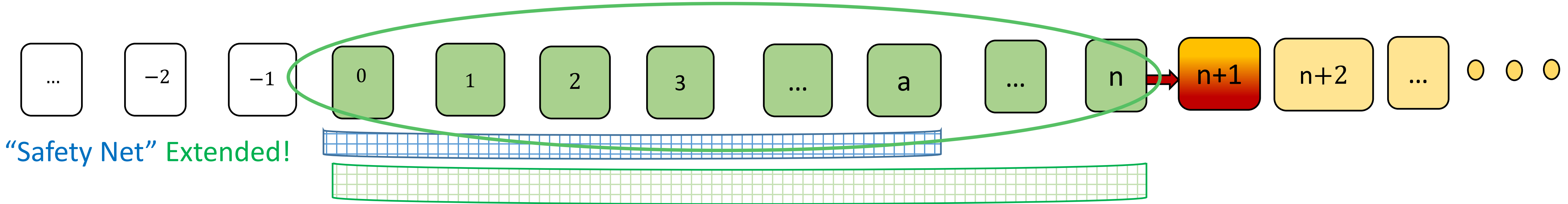

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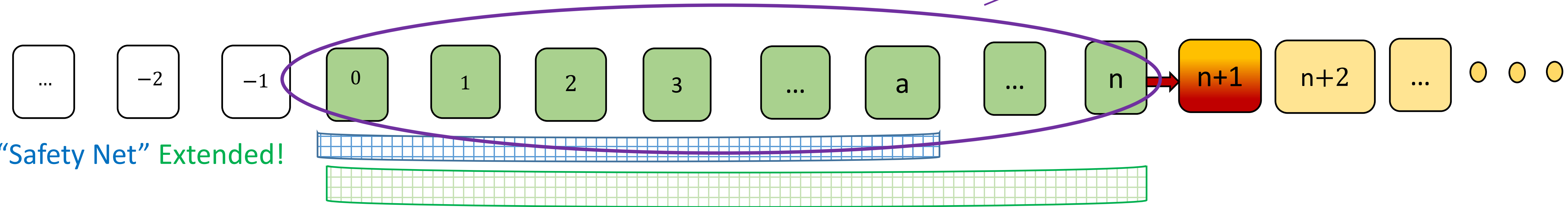
Note that we assume $\boldsymbol{P(0) \wedge P(1) \wedge \cdots \wedge P(n)}$!



"Safety Net" Extended!

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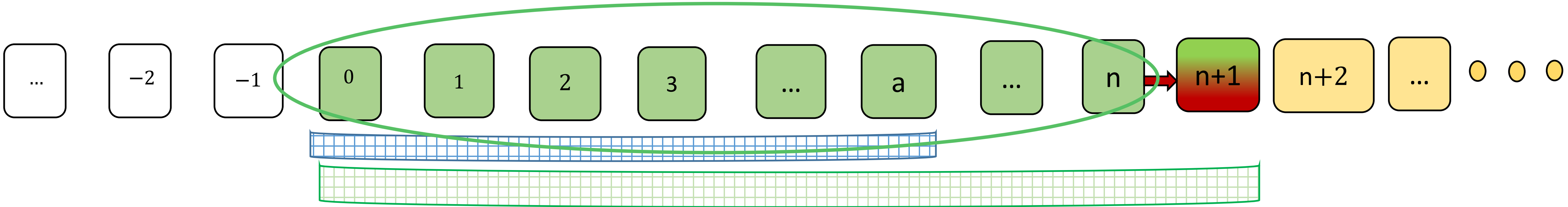

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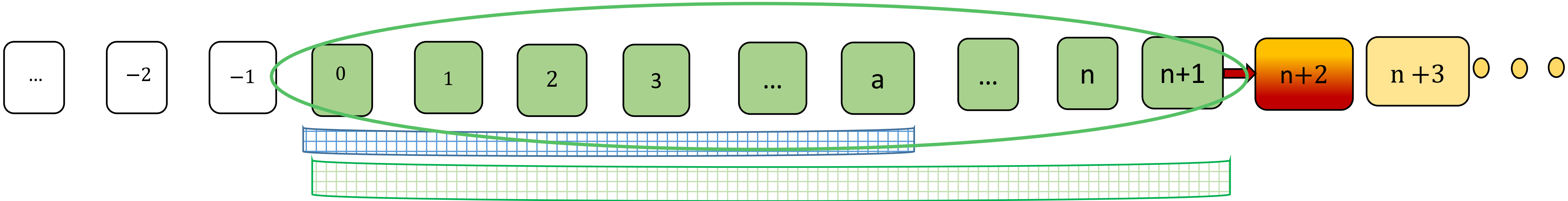

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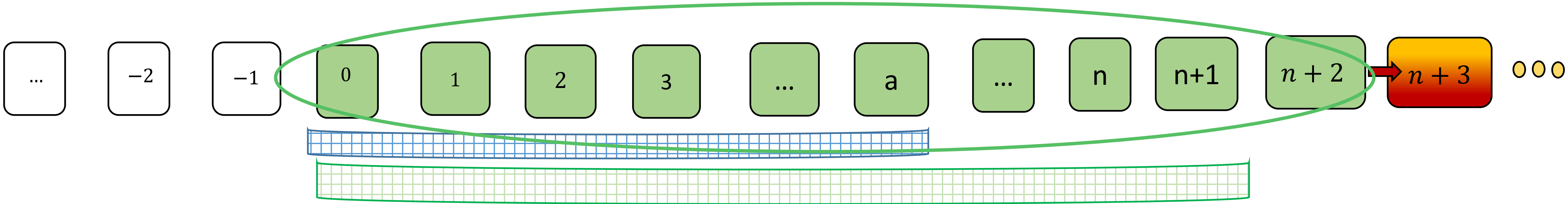

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1 2 3 4

1. Outlining terms
2. Recursive definition
3. Closed-form formula

- Also useful in the study of **algorithm <u>correctness`.</u>**

A First Example

- Let a be a sequence such that:

$$a_n = \begin{cases} 1, & n = 0 \\ 8, & n = 1 \\ a_{n-1} + 2 \cdot a_{n-2}, & n \geq 2 \end{cases}$$

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$$P(n) \Leftrightarrow a_n = 3 \cdot 2^n + 2(-1)^{n+1}$$

- For $n = 0$, $a_0 = 1$ by the definition of a. $P(0)$ says: $a_0 = 3 \cdot 2^0 + 2(-1)^1 = 3 - 2 = 1$.
  So $P(0)$ holds.
- For n = 1, $a_1 = 8$ by the definition of a. $P(1)$ says: $a_1 = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8$.

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Inductive Base established!

Inductive Hypothesis

$P(\boxed{0})$ $P(\boxed{1})$

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- Suppose $n = k \geq 1$. Then, $\forall i \in \{0, 1, \ \dots, n\}$ assume $P(i)$, i.e

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Inductive Hypothesis made!

Inductive Step

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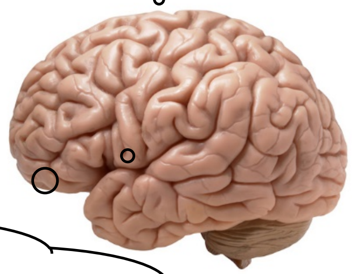

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- Since $\boldsymbol{n \geq 1 \Rightarrow (n+1) \geq 2}$, we can apply the recursive rule of the sequence.
- From the recursive definition of a_n, we obtain:

$$a_{n+1} = a_n + 2 \cdot a_{n-1} \overset{I.H}{=} 3 \cdot 2^n + 2(-1)^{n+1} + 2 \cdot \left(3 \cdot 2^{n-1} + 2(-1)^n\right) =$$
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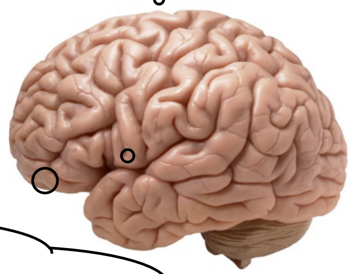

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Inductive step proven!

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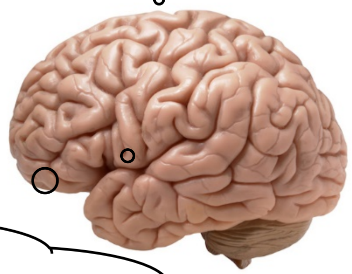

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Proof done!

$\square$

Here's Another

- Suppose that the sequence a_n is as follows:

$$a_n = \begin{cases} 12, & n = 0 \\ 29, & n = 1 \\ 5a_{n-1} - 6a_{n-2}, & n \geq 2 \end{cases}$$

- Then, prove that $a_n = 5 \cdot 3^n + 7 \cdot 2^n, \forall n \in \mathbb{N}$

Inductive Base

- Let the statement to be proven be called $P(n)$. We proceed via strong induction on n.
- **Inductive base:** We want to ***prove*** $P(0), P(1)$.
  - For $n = 0, P(0)$ is $s_0 = 5 \cdot 3^0 + 7 \cdot 2^0 \Leftrightarrow 12 = 12$
  - For $n = 1, P(1)$ is $s_1 = 5 \cdot 3^1 + 7 \cdot 2^1 \Leftrightarrow 29 = 15 + 14$

So the inductive base has been established!

Inductive Hypothesis

- Inductive Hypothesis: Let $n \geq 1$. Then, we ***assume*** that, **for all** $i = 0, 1, \ldots, n, P(i)$ holds, i.e

$$a_i = 5 \cdot 3^i + 7 \cdot 2^i, \quad i = 0, 1, \ldots, n$$

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- Since $(n \geq 1)$, $(n + 1 \geq 2)$ and we can use the recursive definition of a.
- From the recursive definition of a we have:

$$a_{n+1} = 5a_n - 6a_{n-1} \overset{I.H}{=} 5(5 \cdot 3^n + 7 \cdot 2^n) - 6\left(5 \cdot 3^{n-1} + 7 \cdot 2^{n-1}\right)$$
$$= 25 \cdot 3^n + 35 \cdot 2^n - 30 \cdot 3^{n-1} - 42 \cdot 2^{n-1}$$
$$= 5 \cdot (5 \cdot 3^n - 2 \cdot 3^n) + 7(5 \cdot 2^n - 3 \cdot 2^n) = 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1} \quad \square$$

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- Inductive Step: We will attempt to ***prove*** $P(n+1)$, i.e

$$a_{n+1} = 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1}$$

- Since $(n \geq 1)$, $(n+1 \geq 2)$ and we can use the recursive definition of a.
- From the recursive definition of a we have:

$$a_{n+1} = 5a_n - 6a_{n-1} \overset{I.H}{=} 5(5 \cdot 3^n + 7 \cdot 2^n) - 6\left(5 \cdot 3^{n-1} + 7 \cdot 2^{n-1}\right)$$
$$= 25 \cdot 3^n + 35 \cdot 2^n - 30 \cdot 3^{n-1} - 42 \cdot 2^{n-1}$$
$$= 5 \cdot (5 \cdot 3^n - 2 \cdot 3^n) + 7(5 \cdot 2^n - 3 \cdot 2^n) = 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1} \quad \square$$

Since we need factors of 5 and 7 in our result, we force them to appear and our lives automatically become easier!

A Sequence Problem for You!

- Let a_n be defined as:

$$a_n = \begin{cases} 5, & n = 0 \\ 16, & n = 1 \\ 7a_{n-1} - 10a_{n-2}, & n \geq 2 \end{cases}$$

- Prove that $a_n = 3 \cdot 2^n + 2 \cdot 5^n$
- Breakout Rooms

Another Sequence Problem

- Let a_n be defined as:

$$a_n = \begin{cases} 3, & n = 0 \\ 5, & n = 1 \\ 3a_{n-1} - 2a_{n-2}, & n \geq 2 \end{cases}$$

- Prove that $a_n = 2^{n+1} + 1$

Important Note

- In our proofs on recurrences, $P(n+1)$ dependent on stuff such as

$$P(n), P(n-1), P(n-2), \ldots$$

- It is possible (and common) for $P(n+1)$ to depend on

$$P\left({}^{(n+1)}/_{2}\right), P\left({}^{(n+1)}/_{3}\right), P(\sqrt{n+1}) \ldots$$

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STOP RECORDING