START RECORDING

Sequences, Series and Summation / Product Notation

CMSC 250

Sequences and Series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
  - Typical notation: $a: \mathbb{N} \to \mathbb{C}$

Sequences and Series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
  - Typical notation: $a: \mathbb{N} \to \mathbb{C}$
  - Examples:

Sequences and Series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
  - Typical notation: $a: \mathbb{N} \to \mathbb{C}$
  - Examples:
    - 1, 2, 3, 4, 5, …
    - 1.5, 2.5, 3.5, …
    - 1, 1, 1, 1, ….
    - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}$…

Outlining terms

Sequences and Series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
  - Typical notation: $a: \mathbb{N} \to \mathbb{C}$
  - Examples:
    - 1, 2, 3, 4, 5, …
    - 1.5, 2.5, 3.5, …
    - 1, 1, 1, 1, ….
    - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7} \ldots$
    - $a_n = 2^n, n = 0, 1, 2, \ldots$
    - $b_k = logk + 2k, k = 1, 2, 3, \ldots$

Outlining terms

"Closed form" formula

Sequences and Series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
  - Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
  - Examples:
    - 1, 2, 3, 4, 5, …
    - 1.5, 2.5, 3.5, …
    - 1, 1, 1, 1, ….
    - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7} \ldots$
    - $a_n = 2^n, n = 0, 1, 2, \ldots$
    - $b_k = log k + 2k, k = 1, 2, 3, \ldots$
    - $F_n = \begin{cases} 1, & if\ n = 0, 1 \\ F_{n-1} + F_{n-2}, & if\ n \geq 2 \end{cases}$
    - $T_n = \begin{cases} 1, & if\ n = 1,2 \\ 2, & if\ n = 3 \\ T_{n-1} + T_{n-2} + T_{n-3}, & if\ n \geq 4 \end{cases}$

Outlining terms

"Closed form" formula

Recursive formula

Sequences and Series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
  - Typical notation: $a: \mathbb{N} \to \mathbb{C}$
  - Examples:
    - 1, 2, 3, 4, 5, …
    - 1.5, 2.5, 3.5, …
    - 1, 1, 1, 1, ….
    - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}\ldots$
    - $a_n = 2^n, n = 0, 1, 2, \ldots$
    - $b_k = logk + 2k, k = 1, 2, 3, \ldots$
    - $F_n = \begin{cases} 1, & if\ n = 0, 1 \\ F_{n-1} + F_{n-2}, & if\ n \geq 2 \end{cases}$
    - $T_n = \begin{cases} 1, & if\ n = 1,2 \\ 2, & if\ n = 3 \\ T_{n-1} + T_{n-2} + T_{n-3}, & if\ n \geq 4 \end{cases}$

Outlining terms

"Closed form" formula

Recursive formula

All of those are **valid ways** to describe a sequence!

Recursion: Good Idea?

- Example: Fibonacci

$$F_n = \begin{cases} 1\,, & if\ n = 0, 1 \\ F_{n-1} + F_{n-2}\,, & if\ n \geq 2 \end{cases}$$

- We ***can*** use recursion to compute, say, F_{1000}
- Is it a good idea?

Yes

No

Something Else

Recursion: Good Idea?

- Example: Fibonacci

$$F_n = \begin{cases} 1\,, & if\ n = 0, 1 \\ F_{n-1} + F_{n-2}\,, & if\ n \geq 2 \end{cases}$$

- We ***can*** use recursion to compute, say, F_{1000}
- Is it a good idea?

Yes | No | Something Else

- Recomputing terms + hidden memory cost of recursion!

Recursion: Done Right

- Is there a better way to compute F_{1000}?

Yes

No

Something Else

Recursion: Done Right

- Is there a better way to compute F_{1000}?

Yes | No | Something Else

1. Store the values of $F_0 = 1,\ F_1 = 1$ in an array A.
2. for i = 2 to 1000

   $F_i = A[i-1] + A[i-2]$

   $A[i] = F_i$

   end

- This is a very elementary example of a **very useful technique** called ***dynamic programming.***

Closed Formula for Fibonacci

- The closed-form formula for F_n is:

$$F_n = \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\phi}^{n} - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\psi}^{n}$$

- Roughly: $F_n \approx \phi^n \approx (1.618)^n$

Recursion vs Closed Formula

1. Computation:
   - Recursion leads to a fast dynamic program.
   - Classic recursion is elegant.
   - Closed form: faster, but numerical issues arise.
2. Rate of growth:
   - Recursion gives no hint as to **how big** F_n is.
   - Closed form yields $F_n \approx (1.618)^n$

Summation Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their sum, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

$$\sum_{i=1}^{k} a_i$$

Summation Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \ldots, a_k$.
- Their sum, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

$$\sum_{i=1}^{k} a_i$$

Index of last term **to be included** in the sum

Index of term **with which the sum begins**

Examples

$$\sum_{i=1}^{2} a_i = a_1 + a_2$$

$$\sum_{i=1}^{1} a_i = a_1$$

$$\sum_{i=1}^{0} a_i = ?$$

0

1

Something Else

Examples

$$\sum_{i=1}^{2} a_i = a_1 + a_2$$

$$\sum_{i=1}^{1} a_i = a_1$$

$$\sum_{i=1}^{0} a_i = ?$$

0

1

Something Else

$$\sum_{i=1}^{0} a_i = 0$$

- Two reasons for this:
  a) *(Intuitive)* If you add together 0 things, you get 0. Duh.

$$\sum_{i=1}^{0} a_i = 0$$

- Two reasons for this:
  a) *(Intuitive)* If you add together 0 things, you get 0. Duh.
  b) *(Mathematical)* The following formula should work regardless of our choice of integer variable n_1:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n_1} a_i + \sum_{i=n_1+1}^{n} a_i$$

$$\sum_{i=1}^{0} a_i = 0$$

- Two reasons for this:
  a) *(Intuitive)* If you add together 0 things, you get 0. Duh.
  b) *(Mathematical)* The following formula should work regardless of our choice of integer variable n_1:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n_1} a_i + \sum_{i=n_1+1}^{n} a_i$$

So what happens if we pick $n_1 = 0$?

$$\sum_{i=1}^{0} a_i = 0$$

- Two reasons for this:
  a) *(Intuitive)* If you add together 0 things, you get 0. Duh.
  b) *(Mathematical)* The following formula should work regardless of our choice of integer variable n_1:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n_1} a_i + \sum_{i=n_1+1}^{n} a_i$$

So what happens if we pick $n_1 = 0$?
Then, for this to work, it's necessary that $\sum_{i=1}^{0} a_i = 0$

Product Notation

- The **product**, $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is denoted as:

$$\prod_{i=1}^{k} a_i$$

Index of term **with which the product begins** (the lower limit, 1 in $i=1$)

Index of last term **to be included** in the product (the upper limit, k)

$$\prod_{i=1}^{0} a_i = \cdots$$

0	1	Something Else

$$\prod_{i=1}^{0} a_i = \cdots$$

0

1

Something Else

$$\prod_{i=1}^{0} a_i = 1$$

- The following formula has to work for all choices of $n_1 \in \mathbb{N}$:

$$\prod_{i=1}^{n} a_i = \prod_{i=1}^{n_1} a_i \cdot \prod_{i=n_1+1}^{n} a_i$$

- So, for $n_1 = 0$, we need $\prod_{i=1}^{0} a_i = 1$

Sum / Product Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their **sum**, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

Index of last term **to be included** in the sum

$$\sum_{i=1}^{k} a_i$$

Index of term **with which the sum begins**

- Their **product**, $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is denoted as:

Sum / Product Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \ldots, a_k$.
- Their **sum**, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

$$\sum_{i=1}^{k} a_i$$

Index of last term **to be included** in the sum

Index of term **with which the sum begins**

- Their **product**, $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is denoted as:

$$\prod_{i=1}^{k} a_i$$

Index of last term **to be included** in the product

Index of term **with which the product begins**

Sum / Product Notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \ldots, a_k$.
- Their **sum**, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

$$\sum_{r=1}^{k} a_r$$

- Their **product**, $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is denoted as:

$$\prod_{j=1}^{k} a_j$$

"Running" (or "looping" indices can be anything we want! $(i, j, k, \ldots)$ **as long as I use the same variable in the Σ and Π symbols and the variable representing the sequence term!**

Sum-Product Notation

- We can have certain ***exclusionary conditions*** under the Σ and Π symbols.
- Examples:

$$\sum_{\substack{m=0 \\ m \text{ even}}}^{100} s_m$$

$$\sum_{\substack{m=0 \\ 3 \mid m}}^{100} s_m \neq \sum_{\substack{m=0 \\ 3 \mid s_m}}^{100} s_m$$

Series and Partial Sums

- A ***series*** is the **sum** of **all** elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \cdots$$

Or 1, if we start at 1

- A **partial sum** of a sequence, denoted S_n, is the sum ranging from the first up to (and including) the n^{th} term of a (usually infinite) sequence:

$$S_n = \sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \cdots + a_n$$

Or 1, if we start at 1

Famous Sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a_0, \underbrace{a_0 + d}_{\alpha_1}, \underbrace{a_1 + d}_{\alpha_2}, \underbrace{a_2 + d}_{\alpha_3} \ldots \text{ where } \quad d \in \mathbb{R}$$

Famous Sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a_0, \underbrace{a_0 + d}_{a_1}, \underbrace{a_1 + d}_{a_2}, \underbrace{a_2 + d}_{a_3} \ldots \text{ where } \quad d \in \mathbb{R}$$

- Question: which among the following is the correct characterization for a_n?

$d \cdot a_{n-1}$	$a_0 + d \cdot a_{n-1}$	$a_0 + n \cdot d$	$a_0 + (n-1) \cdot d$

Famous Sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a_0, \underbrace{a_0 + d}_{\alpha_1}, \underbrace{a_1 + d}_{\alpha_2}, \underbrace{a_2 + d}_{\alpha_3} \ldots \text{ where } \quad d \in \mathbb{R}$$

- Question: which among the following is the correct characterization for a_n?

$d \cdot a_{n-1}$	$\alpha_0 + d \cdot a_{n-1}$	$\alpha_0 + n \cdot d$	$\alpha_0 + (n-1) \cdot d$

A Question for You

- In the arithmetic progression:

$$a_0, a_0 + d, a_1 + d, a_2 + d \ldots \text{where} \quad d \in \mathbb{R}$$

- Should we allow $d = 0$?

YES NO

A Question for You

- In the arithmetic progression:

$$a_0, a_0 + d, a_1 + d, a_2 + d \ldots \text{where} \quad d \in \mathbb{R}$$

- Should we allow $d = 0$?

It will be a pretty boring sequence, but it will still be a sequence!

YES

NO

Famous Sequences

- **Geometric** sequence (or **progression**):

$$a_0, \underbrace{m \cdot a_0}_{\alpha_1}, \underbrace{m \cdot a_1}_{\alpha_2}, \underbrace{m \cdot a_2}_{\alpha_3} \ldots$$

Famous Sequences

- **Geometric** sequence (or **progression**):

$$a_0, m \cdot a_0, m \cdot a_1, m \cdot a_2 \ldots, \qquad m \in \mathbb{R}$$

α_1 α_2 α_3

- Question: which among the following is the correct characterization for a_n?

$(m-1)^n \cdot a_0$ | $m^n a_0$ | $m^{n-1} a_0$ | $m \cdot n \cdot a_0$

Famous Sequences

- **Geometric** sequence (or **progression**):

$$a_0, m \cdot a_0, m \cdot a_1, m \cdot a_2 \ldots, \qquad m \in \mathbb{R}$$

α_1 α_2 α_3

- Question: which among the following is the correct characterization for a_n?

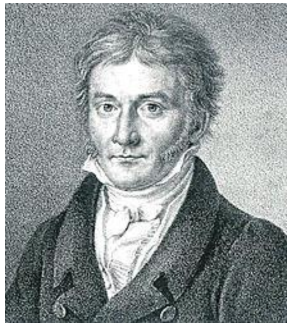
$(m-1)^n \cdot a_0$

$m^n a_0$

$m^{n-1} a_0$

$m \cdot n \cdot a_0$

The Gauss Story

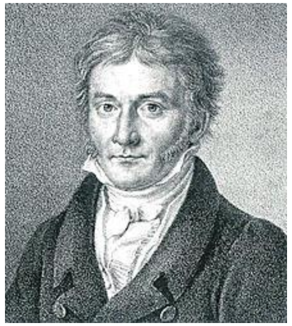


- Gauss was a great mathematician (1777-1855)
- When Gauss was in 1st grade, the class was misbehaving.
- For punishment, the teacher made everyone compute

$$1 + 2 + \cdots + 100$$

- Gauss did it in 2 minutes. Can you?

The Gauss Trick



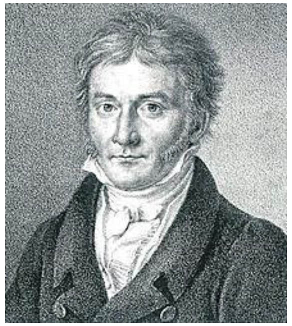
$$S = 1 + 2 + \cdots + 100$$
$$S = 100 + 99 + \cdots + 1$$

$$+$$

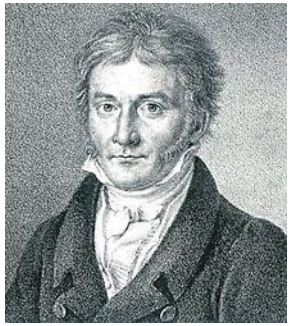
$$2S = \underbrace{101 + 101 + \cdots + 101}_{\text{100 terms}}$$

$$\Rightarrow 2S = 101 * 100 = 10100 \Rightarrow S = 5050$$

And Now the Rest of the Story

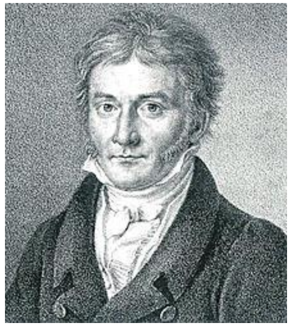


And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

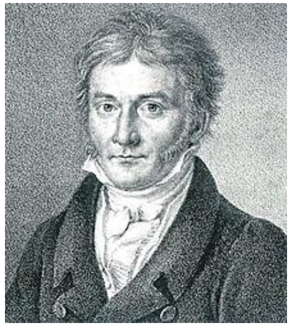
And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES

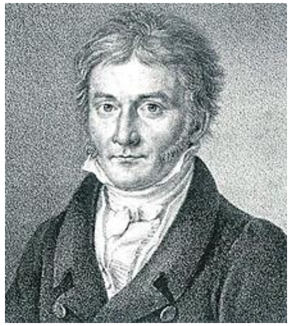
And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5^{th}	1 + 2 + … + 60

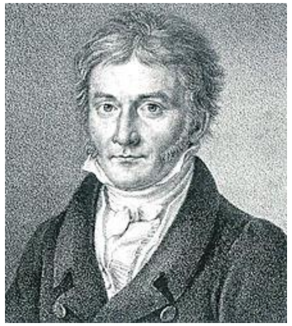
And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5^{th}	1 + 2 + … + 60
1980	3^{rd}	1 + 2 + … + 80

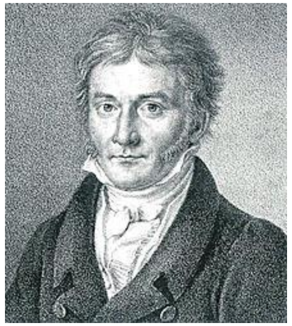
And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5^{th}	1 + 2 + … + 60
1980	3^{rd}	1 + 2 + … + 80
2000s	1^{st}	1 + 2 + … + 100

And Now the Rest of the Story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5th	1 + 2 + … + 60
1980	3rd	1 + 2 + … + 80
2000s	1st	1 + 2 + … + 100
2020	Nursery School	1 + 2 + … + 120

Our conjecture: (the 2020 row)

Famous Sequences

- **Harmonic**:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- **Fibonacci**: $F_0 = F_1 = 1$ and $\forall n \geq 2, F_n = F_{n-1} + F_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21, …

What We'll Do Next

- We will have an intro to **induction.**
- The following can be proven via induction:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

STOP
RECORDING