START

RECORDING

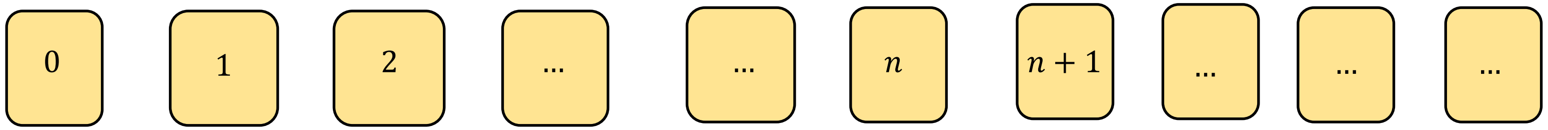
Mathematical Induction: Introduction and Basic Problems

CMSC 250

INTRO AND BASIC SEQUENCE PROBLEMS

The Idea Behind Induction

- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n.



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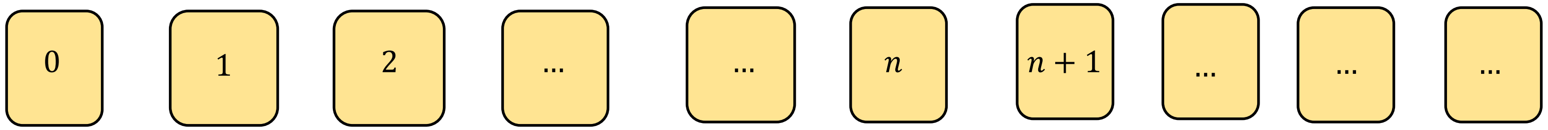
- We will prove two separate things:



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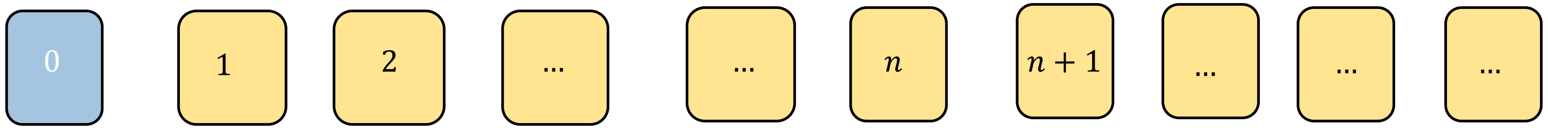
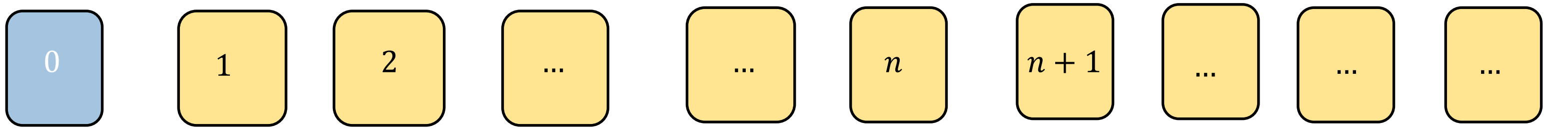
- We will prove two separate things:
  1. For $n = 0$, $P(n)$ is true



The Idea Behind Induction

- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n.
- We will prove two separate things:
  1. For $n = 0$, $P(n)$ is true *(simplifiable to "$P(0)$ is true")*.



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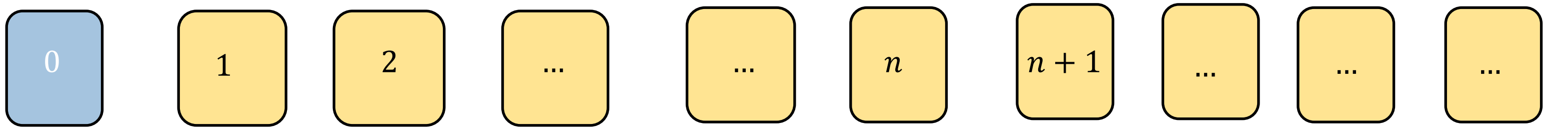
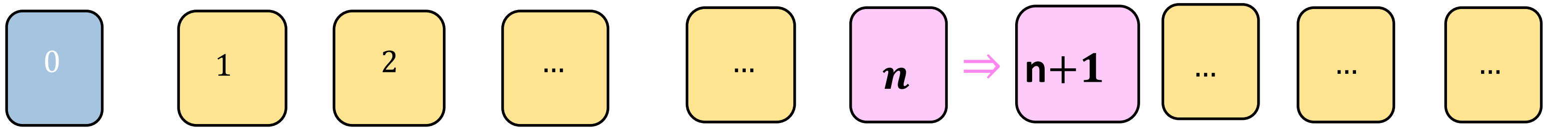


The Idea Behind Induction

- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n.
- We will prove two separate things:
  1. $P(0)$ is true.
  2. For all $n \geq 1, P(n) \Rightarrow \mathrm{P(n+1)}$



How We'll Make It Work

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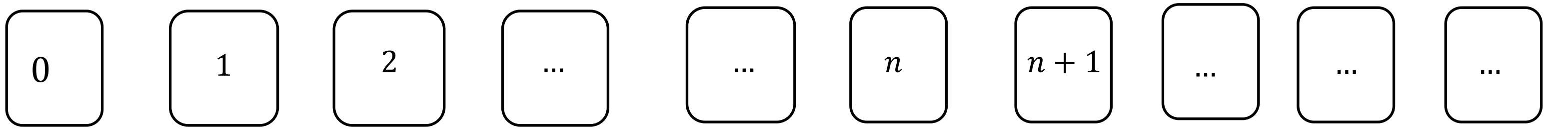

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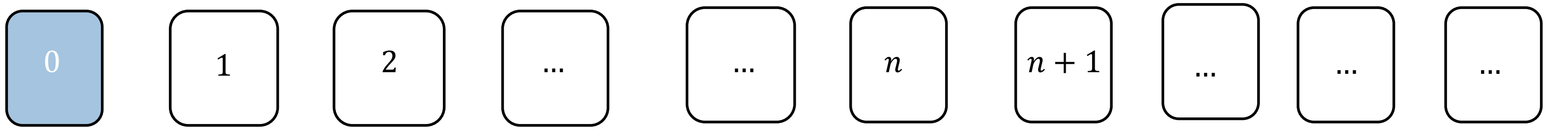

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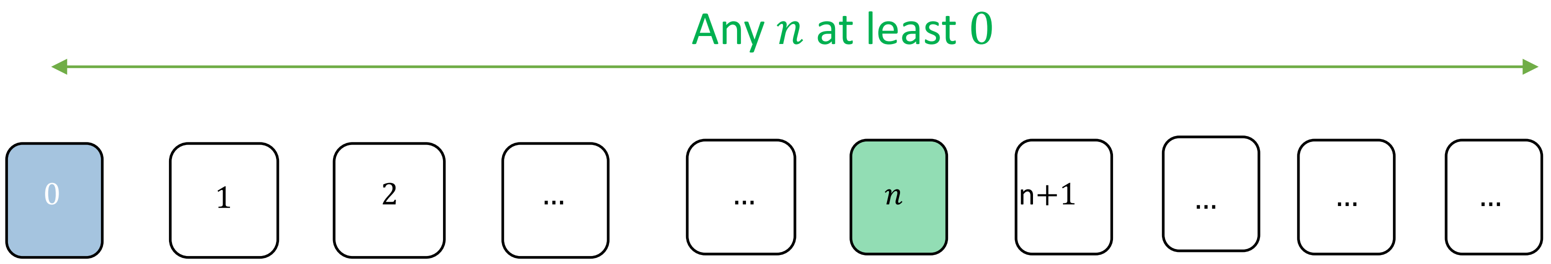

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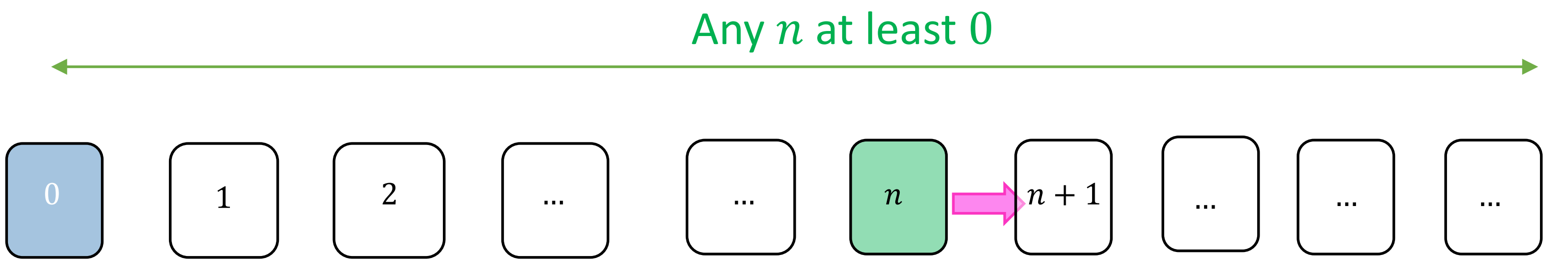

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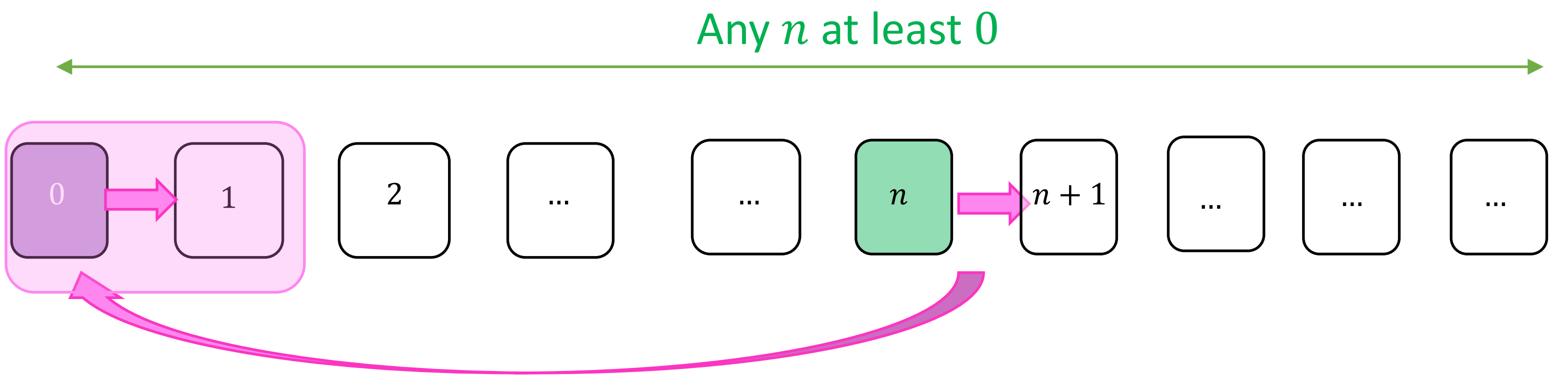

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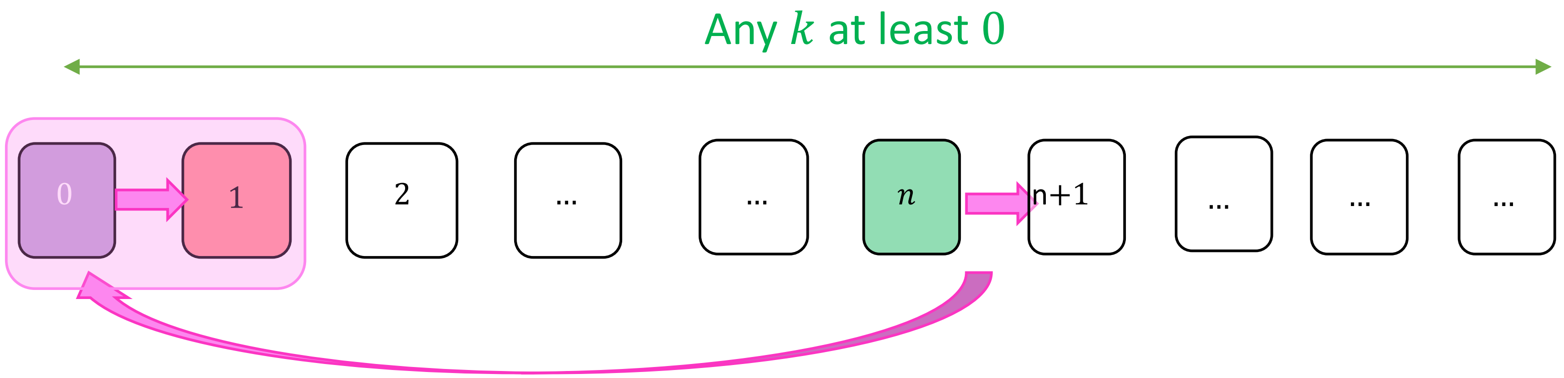

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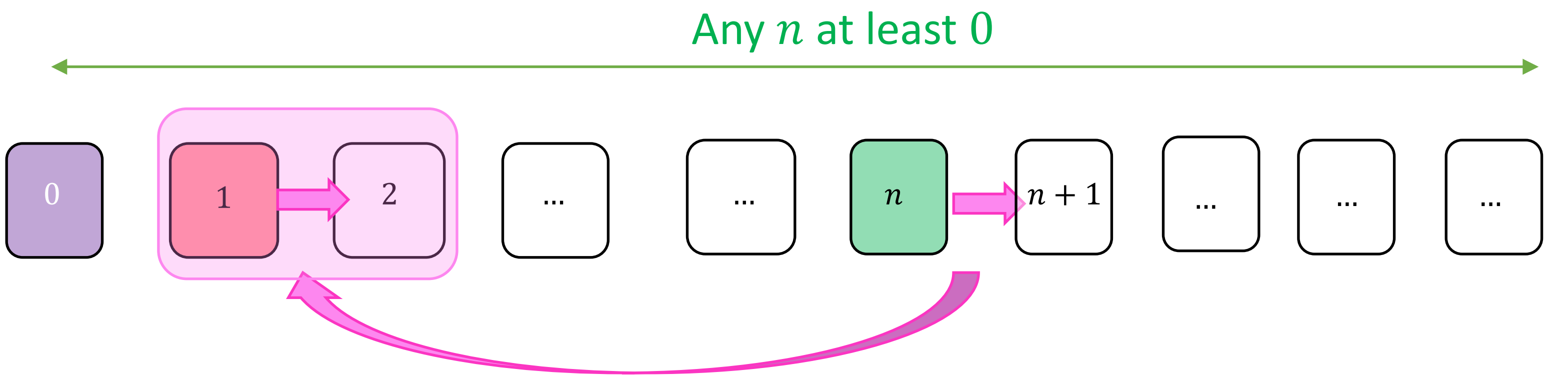

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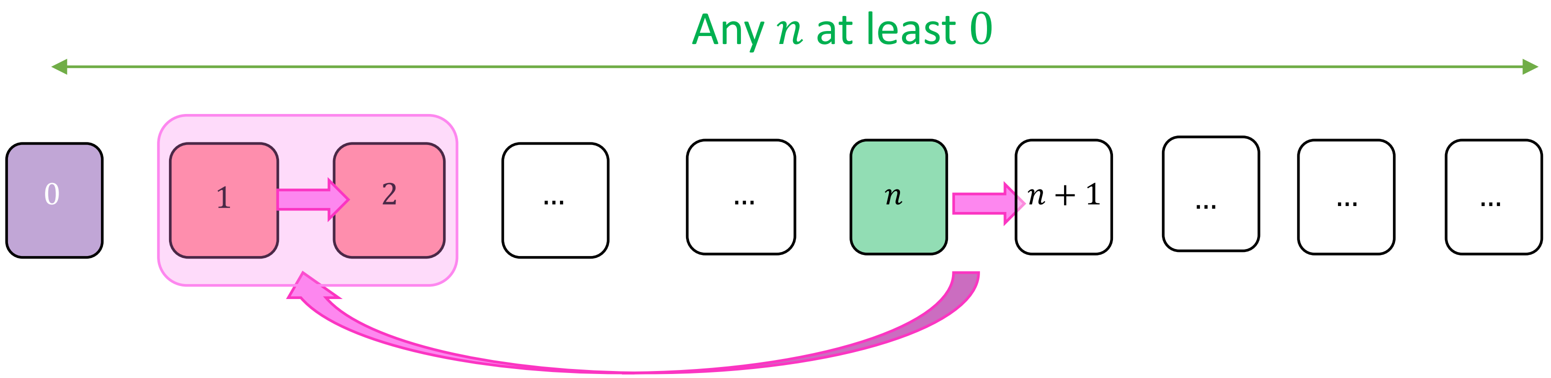

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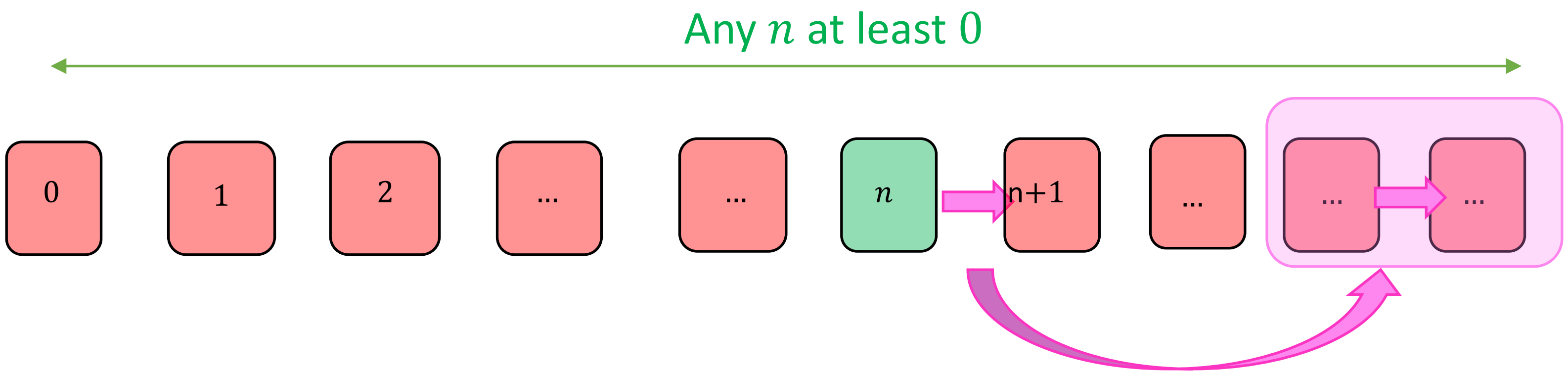

(We fast-forwarded here to save some time.)

SUM PROBLEMS

$$\sum_{i=0}^{n} f(n)$$

The Gaussian Sum

- We will prove that the sum of the first n numbers is equal to $\frac{n(n+1)}{2}$.
- Symbolically:

$$\underbrace{1 + 2 + 3 + \cdots + (n - 1) + n}_{\sum_{i=1}^{n} i} = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Inductive Base

- For $n = 0$, we will **prove** that $P(0)$ holds

$$\sum_{i=1}^{0} i \overset{?}{=} \frac{0(0+1)}{2}$$

Remember: $P(n)$ is
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- LHS: $\sum_{i=1}^{0} i = 0$ (recall this fact from our sequences lecture)
- RHS: $\frac{0(0+1)}{2} = 0$
- Since LHS = RHS for $n = 0$, $P(0)$ has been proven true.

Inductive Hypothesis

- For $n \geq 0$, we **assume** that $P(n)$ **is true**:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

So, we **assume** that

$$P(n) \Leftrightarrow \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is true for an arbitrary n≥ 0

- Inductive Hypothesis done!

Inductive Step

- Given that $P(n)$ is true, we will **prove** that $P(n+1)$ is true.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

Inductive Step

- Given that $P(n)$ is true, we will **prove** that $P(n+1)$ is true.

Just adding 1 to n

Just adding 1 to n+1

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

Inductive Step

- Given that $P(n)$ is true, we will **prove** that $P(n+1)$ is true.

Just adding 1 to n

Just adding 1 to n+1

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

This is our goal!

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1)$$

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1) = \sum_{i=1}^{n} i + (n+1) \qquad (1)$$

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1) = \sum_{i=1}^{n} i + (n+1) \quad (1)$$

- **From the Inductive Hypothesis**, we have that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad (2)$$

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1) = \sum_{i=1}^{n} i + (n+1) \quad (1)$$

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Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

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$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad (2)$$

- Substituting (2) into (1) yields (next slide):

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2}$$

$$= RHS$$

Inductive Step, contd.

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2}$$

$$= RHS$$

- So, when $P(n)$ is true, $P(n+1)$ was also proven true.
- We conclude that $P(n)$ is true $\forall n \geq 0$.
- WE ARE DONE.

Here's Another!

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

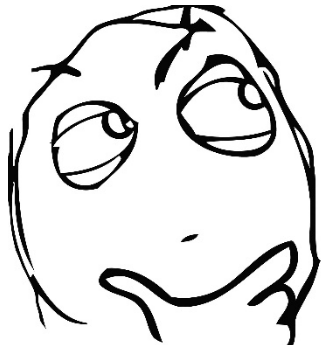
Inductive Base

- For $n = 0$, LHS = $\sum_{i=1}^{0} i^2 = 0$
- RHS=$\frac{0(0+1)(2*0+1)}{2} = 0$
- Since LHS = RHS, $P(0)$ holds and we are done.

Inductive Base

- For $n = 0$, LHS = $\sum_{i=1}^{0} i^2 = 0$
- RHS=$\frac{0(0+1)(2*0+1)}{2} = 0$
- Since LHS = RHS, $P(0)$ holds and we are done.

- You could also start from $n = 1$! LHS = RHS in both cases
  - $n = 0$ sometimes makes the math easier (RHS in this case)



Inductive Hypothesis

- Suppose that $n \geq 0$. *(Or 1 in the alternative scenario)*
- We will then assume $P(n)$, i.e:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Inductive Step

- We will now attempt to prove $P(n+1)$, i.e

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Careful with factoring please!!!

Inductive Step

- We will now attempt to prove $P(n+1)$, i.e

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Careful with factoring please!!!

- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

Inductive Step

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- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

We can apply the IH here!

Inductive Step

- By IH, we can now write:

$$\sum_{i=1}^{n+1} i^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

- Remember: we **want** this to be equal to

$$\frac{(n+1)(n+2)(2n+3)}{6}$$

- **We will <u>fearlessly</u> manipulate the algebra <u>until it does</u>!**

Inductive Step - Algebra

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}$$

$$= \frac{(n+1)[n(2n+1)+6(n+1)]}{6} = \frac{(n+1)[2n^2+7n+6]}{6}$$

- If only we could prove that $2n^2 + 7n + 6 = (n+2)(2n+3)$, we'd be done!
- But.... $(n+2)(2n+3) = 2n^2 + 3n + 4n + 6 = 2n^2 + 7n + 6$! ☺
- So we're done.

Sums of Powers of 2

- Prove that the sum of the first n terms of a **geometric sequence** with $a_1 = 1$ is equal to $2^n - 1$.
- Symbolically:

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Proof

- Proof : We attempt to prove $P(n),\ \forall n \in \mathbb{N}$. We proceed via induction on n.
- **Inductive base**: We attempt to prove $P(0)$.

$$P(0): \sum_{i=0}^{1-1} 2^i = 2^0 - 1 \Leftrightarrow 0 = 0$$

So $P(0)$ is true.

- **Inductive hypothesis**: Suppose $n \geq 0$. We assume $P(n)$, i.e

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Proof (contd.)

- **Inductive step:** We will attempt to prove $P(n+1)$, i.e

$(n+1)-1$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

From the LHS to the RHS:

By the IH

$$LHS = \sum_{i=0}^{n} 2^i = \sum_{i=0}^{n-1} 2^i + 2^n = 2^n - 1 + 2^n = 2(2^n) - 1 = 2^{n+1} - 1 = RHS \; \square$$

Sums of Powers of m

- Prove that the sum of the first n terms of a **geometric sequence** with $m \in (\mathbb{R} - \{1\})$ and $a_1 = 1$ is equal to $\frac{m^n - 1}{m - 1}$.
- Symbolically:

$$\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}$$

Proof

- Proof : We attempt to prove $P(n), \ \forall n \in \mathbb{N}$. We proceed via induction on n.
- **Inductive base**: We attempt to prove $P(0)$.

$$P(0): \sum_{i=0}^{1-1} m^i = \frac{m^1 - 1}{m - 1} \Leftrightarrow \sum_{i=0}^{0} m^i = \frac{m^1 - 1}{m - 1} \Leftrightarrow 1 = 1$$

So $P(0)$ is true.

Note: In the base case we are assuming $m \neq 1$

- **Inductive hypothesis**: Suppose $n \geq 0$. We assume $P(n)$, i.e

$$\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}$$

Proof (contd.)

- **Inductive step:** We will attempt to prove $P(n+1)$, i.e

$$\sum_{i=0}^{n} m^i = \frac{m^{n+1} - 1}{m - 1}$$

$(n+1)-1$

From the LHS to the RHS:

By the IH

$$LHS = \sum_{i=0}^{n} m^i = \sum_{i=0}^{n-1} m^i + m^n = \frac{m^n - 1}{m-1} + m^n = \frac{m - 1 + m^n(m-1)}{m-1} = \frac{m^{n+1}-1}{m-1} = RHS \ \square$$

Base Cases

- It is standard to change your base cases to later in your index if the theorem you are trying to prove starts later

COIN PROBLEMS!

A Coin Problem

- We will prove that every dollar amount $\geq$ 4 cents can be exclusively paid for by 2 and/or 5 cent coins.

Theorem Expressed in Quantifiers

- All quantifiers implicitly assumed over $\mathbb{N}$.

$$(\forall n \geq 4)(\exists n_1, n_2)[n = 2n_1 + 5n_2]$$

Inductive Base

- The least amount of money we are required to prove the statement for is 4¢, so we will attempt to prove $P(4)$.
- For $n = 4$, we have 4¢. Since $4¢ = 2 \times 2¢$, we are done (we have shown that the amount of 4¢ can be exclusively paid for by using only 2 and/or 5 cent coins)

Inductive Hypothesis 

- Let n$\geq$ 4.
- Assume $P(n) \Leftrightarrow (\exists n_1, n_2)[n = 2n_1 + 5n_2]$

Inductive Step

- We will prove that $P(n) \Rightarrow P(n+1)$, i.e that we can pay an amount of money equal to $n+1$ cents using **only** 2¢ or 5¢ coins.
- In terms of algebra, what we want to prove is:

$$(\exists n_3, n_4 \in \mathbb{N})\ [n + 1 = 2n_3 + 5n_4]$$

Inductive Step

- We will prove that $P(n) \Rightarrow P(n+1)$, i.e that we can pay an amount of money equal to n+1 cents using **only** 2¢ or 5¢ coins.
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$$(\exists n_3, n_4 \in \mathbb{N})\ [n + 1 = 2n_3 + 5n_4]$$

Different variables from IH!

Inductive Step (contd.)

- From the Inductive Hypothesis (IH), we have that for some specific positive integers n_1 and n_2:

$$n = 2n_1 + 5n_2$$

Inductive Step (contd.)

- From the Inductive Hypothesis (IH), we have that for some specific positive integers n_1 and n_2:

$$n = 2n_1 + 5n_2$$

1. **Case #1:** $n_1 \geq 2$

- I have at least two 2¢ coins, so I can take away two 2¢ coins and add one

  5¢ coin

Inductive Step (contd.)

- From the Inductive Hypothesis (IH), we have that for some specific positive integers n_1 and n_2:

$$n = 2n_1 + 5n_2$$

1. **Case #1:** $n_1 \geq 2$

- I have <u>at least two</u> 2¢ coins, so I can take away two 2¢ coins and add one 5 ¢ coin
- By adding 1 on both sides of the IH we obtain:

$$n + 1 = 2n_1 + 5n_2 + 1 = 2n_1 + 5n_2 + (5 - 2 * 2) =$$
$$= (2n_1 - 4) + (5n_2 + 5) = 2\underbrace{(n_1 - 2)}_{n_3} + 5\underbrace{(n_2 + 1)}_{n_4} = 2n_3 + 5n_4$$

Inductive Step (contd.)

- From the Inductive Hypothesis (IH), we have that for some specific positive integers n_1 and n_2:

$$n = 2n_1 + 5n_2$$

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- By adding 1 on both sides of the IH we obtain:

$$n + 1 = 2n_1 + 5n_2 + 1 = 2n_1 + 5n_2 + (5 - 2 * 2) =$$

$$= (2n_1 - 4) + (5n_2 + 5) = 2\underbrace{(n_1 - 2)}_{n_1 - 2 \geq 0 \text{ because } n_1 \geq 2} + 5\underbrace{(n_2 + 1)}_{\text{In } \mathbb{N} \text{ by closure}} = 2n_3 + 5n_4$$

Inductive Step

2. **Case #2:** $n_2 \geq 1$

- I have <u>at least</u> one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the IH we obtain:

$$n + 1 = 2n_1 + 5n_2 + 1 = 2n_1 + 5n_2 + (3 * 2 - 5) =$$

$$= 2\underbrace{(n_1 + 3)}_{n_3} + 5\underbrace{(n_2 - 1)}_{n_4} = 2n_3 + 5n_4$$

Inductive Step

2. **Case #2:** $n_2 \geq 1$

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the IH we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 * 2 - 5) =$$

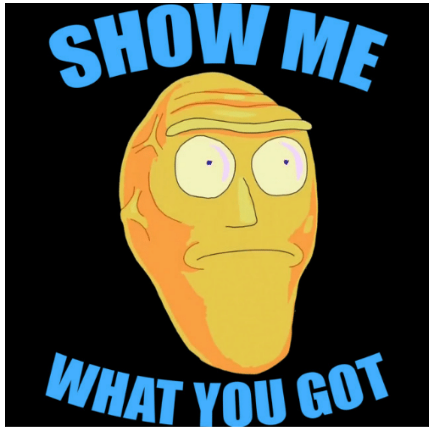
$$= 2\underbrace{(n_1 + 3)}_{\substack{(n_1 + 3) \in \mathbb{N} \\ \text{by closure}}} + 5\underbrace{(n_2 - 1)}_{\substack{n_2 - 1 \geq 0 \\ \text{because} \\ n_2 \geq 1}} = 2n_3 + 5n_4$$

Inductive Step

3. Case #3: $(n_1 \leq 1) \wedge (n_2 = 0)$

- This case means that we have either 0 or 2¢ at our disposal.
- But this is not possible, since we want to prove the theorem only for values ≥ 4¢
- So we're done. □

A Coin Problem for **You!**



Prove to me that every dollar amount $\geq$ 20 cents can be **exclusively** paid for through combinations of **5**-cent coins and **6**-cent coins!

Go to Breakout Rooms

TREATING INEQUALITIES

Here's One with an Inequality!

- Prove that for all integers n at least 4, $2^n < n!$

1. **IB:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **IH:** For $n \geq 4$, we assume $P(n)$, i.e $2^n < n!$
3. **IS:** We will prove $P(n) \Rightarrow P(n+1)$, i.e

$$(2^n < n!) \Rightarrow (2^{n+1} < (n+1)!)$$

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **IB:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **IH:** For $n \geq 4$, we assume $P(n)$, i.e $2^n < n!$
3. **IS:** We will prove $2^{n+1} < (n+1)!$

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   - Since n ≥ 4, we have that $2 < n + 1 \overset{n!>0}{\Longleftrightarrow} n! \cdot 2 < n!\,(n+1)$ (3)

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   - Since $n \geq 4$, we have that $2 < n + 1 \overset{n!>0}{\Longleftrightarrow} n! \cdot 2 < n!\,(n+1)$ (3)
   - $(2) \overset{(3)}{\Rightarrow} 2^n \cdot 2 < (n+1)! \overset{(1)}{\Leftrightarrow} 2^{n+1} < (n+1)!$

STOP RECORDING