**Untimed Midterm. Morally Due April 11**
**WARNING: THIS MIDTERM IS FOUR PAGES LONG!!!!!!!!!!!!!!!!!**

1. (16 points)

   (a) (0 points) In this problem $\mathsf{C}$ is the complex numbers.
   Write a program that does the following: Given $A, B$ consider the recurrence:
   $a_0 = 1$
   $a_1 = 2$
   $(\forall n \ge 2)[a_n = Aa_{n-1} + Ba_{n-2}]$.
   FIND $\alpha_1, \alpha_2 \in \mathsf{C}$ (use an approximation to 5 places) such that there exists $C, D$ and

   $$a_n = C\alpha_1^n + D\alpha_2^n.$$

   (You can find $C, D$ if you want to but they are not required for this problem.)

   (b) (0 points but you will need this) Write a program that will, given $M$, run the program in part a for all $1 \le A \le M$ and $-M \le B \le M$ and generates a table of the following form:
   $M = 2$:

| $A$ | $B$ | $\alpha_1$ | $\alpha_2$ | $\max\{\alpha_1, \alpha_2\}$ |
|---|---|---|---|---|
| 1 | $-2$ | $1.2 + i$ | $2.3 - i$ | $2.3 - i$ |
| 1 | $-1$ | 2.2 | 4.3 | 4.3 |
| 1 | 0 | 8.2 | 1.3 | 8.2 |
| 1 | 1 | 9.2 | 11.3 | 11.3 |
| 1 | 2 | 19.2 | 111.3 | 111.3 |
| 2 | $-2$ | 1.2 | 2.3 | 2.3 |
| 2 | $-1$ | 2.2 | 4.3 | 4.3 |
| 2 | 0 | 8.2 | 1.3 | 8.2 |
| 2 | 1 | 9.2 | 11.3 | 11.3 |
| 2 | 2 | 19.2 | 111.3 | 111.3 |

   (for complex number $a + bi$ the size is $a^2 + b^2$. We use this for defining the max.)

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(c) (0 points) Email Emily your code.

(d) (5 points) IF you ran the code on $M$, how many rows will the program generate? Show your work in deducing the number.

(e) (11 points) Run the code on $M = 3$ and submit the table.

(f) (Extra Credit) Say something intelligent about how $A$ affects MAX ALPHA and how $B$ affects MAX ALPHA. Which has a bigger effect?

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2. (16 points) Emily might *teach* 250H in Spring 2023 (Bill is going on sabbatical). She will need help designing problems! In this problem you will help her!

   She wants to ask a question of the following form (With $A, B, C$ replaced by positive natural numbers).

   HERE IS THE PROBLEM SHE WANTS TO ASK:

   *Let $a_n$ be defined as follows.*

   $a_1 = 5$

   $(\forall n \geq 2)[a_n = Ba_{n-1}^2 + Ca_{\lfloor n^{1/3} \rfloor}]$

   *Show by strong induction that*

   $(\forall n \geq 1)[a_n \equiv 5 \pmod{12}]$

   *Include Base Case, IH, and IS.*

   Now for YOUR PROBLEM: Use constructive induction to find 9 pairs $(B, C)$ such that

   $$(\forall n \geq 1)[a_n \equiv 5 \pmod{12}].$$

   You will need to have a Base Case, IH, and IS.

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3. (18 points- 6 points each) In this problem all of the $x_i$ are natural numbers. And remember that 0 is a natural number.

   (a) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq 0) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$

   (b) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq 1) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$

   (c) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq 2) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$

   (d) (Extra Credit) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq i) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$