

Final MORALLY Due May 9 at 9:00AM
MANY OF THE PROBLEMS ARE WORTH 0 POINTS
THEY ARE TO WRITE SHORT PROGRAMS
THERE WILL BE A PROBLEM WORTH REAL POINTS THAT USES THEM

1. (0 points but please DO IT) What is your name? What day is the timed final?
2. (0 points BUT YOU REALLY NEED TO DO THIS TO UNDERSTAND AND TO THE NEXT TWO PROBLEMS ON THIS HW) Read and understand the notes on THE HALF METHOD on the course websites of class slides.
3. (0 points) For all programs in this problem I specify the output. You may need to modify that part when you use a program as a subroutine in another program.

In this problem and the next we will guide you through the writing of several short programs which you will later put together. The final goals are:

- A program that will, given (m, s, α) determine if $f(m, s) \leq \alpha$. It will try both the Floor-ceiling method (henceforth FC) and the HALF method. It may return that NEITHER method worked.
 - A program that will, given (m, s) determine the smallest α for which $f(m, s) \leq \alpha$ can be proven by either FC or HALF. (This will be done in the next problem though it will draw on this problem.)
- (a) *CHECK-Program*. Input is (m, s, α) . Check that $m > s$ and $\alpha > \frac{1}{3}$. If either is false then output *BAD INPUT*. If not then output *GOOD INPUT*. When writing the programs below separately they should all begin with CHECK. When you put them together into a big program, the big program should begin with CHECK.
 - (b) *FC-Program*. Input (m, s, α) .
If $f(m, s) \leq \alpha$ follows from FC then output *FC establishes bound*.
If not then output *FC does not establish bound*.

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(c) *V-Program*. Input (m, s, α) .

We begin doing the HALF method.

We need to know $V \in \mathbb{N}$ such that

- NOBODY has $\geq V + 1$ pieces. If someone has $\geq V + 1$ pieces than some piece is $\leq \frac{m}{s} \frac{1}{V+1}$. Hence we need $\frac{m}{s} \frac{1}{V+1} \leq \alpha$.
- NOBODY has $\leq V - 2$ pieces. If someone has $\leq V - 2$ pieces than some piece is $\geq \frac{m}{s} \frac{1}{V-2}$. That pieces buddy is $\leq 1 - \frac{m}{s} \frac{1}{V-2}$. Hence we need $\frac{m}{s} \frac{1}{V+1} \leq \alpha$.

If there exists such a V then output *Use V for HALF*.

If not then output *No V exists, so HALF won't work*.

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(d) *Equations-Program.* Input is (m, s, α, V) where V is the output from the V -program. Let

- s_{V-1} be the number of students who get $V - 1$ shares.
- s_V be the number of students who get V shares.

Since there are $2m$ pieces

$$(V - 1)s_{V-1} + Vs_V = 2m$$

Since there are s students

$$s_{V-1} + s_V = s$$

Solve this system of 2 equations in two variables to output s_{V-1} and s_V .

(Note that this step did not require α .)

If $s_{V-1}, s_V \in \mathbb{N}$ and are ≥ 2 then Output (s_{V-1}, s_V) .

If not then output (s_{V-1}, s_V) *out of bounds, so HALF won't work.*

Advice Solve them on paper yourself and then just use those formulas.

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- (e) *V-share-Intervals-Program*. Input is $(m, s, \alpha, V, s_{V-1}, s_V)$ where V comes from the V -program and (s_{V-1}, s_V) comes from the Equations-Program. (We don't actually use s_{V-1} .)

We want to find the intervals for the V -shares.

The V -shares will be in (α, β) for some β that we want to find.

We derive β by contradiction. Assume there is some V -share of size $\geq \beta$. Assume Alice has that share and of course $V - 1$ other shares. call those shares $p_1 < \dots < p_V$. Then

$$\sum_{i=1}^V p_i = \frac{m}{s}$$

$$\sum_{i=1}^{V-1} p_i = \frac{m}{s} - p_V \leq \frac{m}{s} - \beta$$

Then

$$p_1 \leq \frac{(m/s) - \beta}{V - 1}$$

But since $p_1 > \alpha$ we set

$$\left(\frac{(m/s) - \beta}{V - 1} \right) < \alpha$$

$$\frac{m}{s} - \beta < \alpha(V - 1)$$

$$\beta > \frac{m}{s} - \alpha(V - 1)$$

Hence we take $\beta = \frac{m}{s} - \alpha(V - 1)$.

If $\alpha < \beta < 1 - \alpha$ then output β .

If not then output β *does not work, HALF method fails*.

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- (f) *(V - 1)-share-Intervals-Program.* Input is $(m, s, \alpha, V, s_{V-1}, s_V)$ where V comes from the V -program and (s_{V-1}, s_V) comes from the Equations-Program. (We don't actually use s_V .)

We want to find the intervals for the $V - 1$ -shares.

The V -shares will be in $(\gamma, 1 - \alpha)$ for some γ that we want to find.

We derive γ by contradiction. Assume there is some $V - 1$ -share of size $\leq \gamma$. Assume Alice has that share and of course $V - 2$ other shares. Call those shares $p_1 < \dots < p_{V-1}$. Then

$$\sum_{i=1}^{V-1} p_i = \frac{m}{s}$$

$$\sum_{i=2}^{V-1} p_i = \frac{m}{s} - p_1 \geq \frac{m}{s} - \gamma$$

THE REST I LEAVE TO YOU. KEY: IF YOU FIND SOME SHARE IS LARGE, ITS BUDDY IS SMALL.

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(g) *VHALF-Program*. Input is $(m, s, \alpha, V, s_{V-1}, s_V, \beta, \gamma)$ all from the prior programs. SO, what do we know at this point?

- If $\beta > \gamma$ then output NOT GOING TO WORK (or something like that).
- The V -shares are in (α, β) . Hence there are Vs_V shares in (α, β) .
- The $(V - 1)$ -shares are in $(\gamma, 1 - \alpha, \beta)$. Hence there are $(V - 1)s_{V-1}$ shares in $(\gamma, 1 - \alpha)$.

IF $\beta \leq \frac{1}{2} < \gamma$ AND $Vs_V \neq (V - 1)s_{V-1}$ then there are too many shares on one side of $\frac{1}{2}$ and that's a contradiction.

If that is the case then output *HALF method worked!*

IF not then output *HALF method did not work*.

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- (h) *FINAL-FC-VHALF-Program*. Input is (m, s, α) .
- Run FC-program. If it verifies $f(m, s) \leq \alpha$ then output $f(m, s) \leq \alpha$ by the FC theorem.
 - If not then run all of the programs above up to an including VHALF.
If VHALF verifies $f(m, s) \leq \alpha$ then output $f(m, s) \leq \alpha$ by the *HALF* theorem with parameters $(V, s_{V-1}, s_V, \beta, \gamma)$.
 - If neither one verifies then output $f(m, s) \leq \alpha$ cannot be proven by *FC* or *HALF*.

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4. The following are true facts:

- When the HALF method works then $V = \frac{2m}{2}$.
- When the HALF method works either $\beta = \frac{1}{2}$ or $\gamma = \frac{1}{2}$.

We use these two facts to DERIVE α FROM (m, s) .

- (a) *EASY-V-Program*. On input (m, s) output $V = \frac{2m}{s}$.
- (b) *Equations-Program*. On input (m, s, V) output s_V and s_{V-1} . Same as in Problem 3 since, as we noted, the *Equations Program* uses (m, s, V) but not α .
- (c) $\beta = \frac{1}{2}$ -*Program*. Input (m, s, V) Recall that in the program in Problem 3 we had that the V -shares were in (α, β) where

$$\beta = \frac{m}{s} - \alpha(V - 1).$$

But this time we are going to *assume* $\beta = \frac{1}{2}$. With that in mind output what α is. Call it α_1 .

- (d) $\gamma = \frac{1}{2}$ -*Program*. Input (m, s, V) Recall that in the program in Problem 3 we had that the $(V - 1)$ -shares were in $(\gamma, 1 - \alpha)$ where I LEFT γ to YOU to figure out.

But this time we are going to *assume* $\gamma = \frac{1}{2}$. With that in mind output what α is. Call it α_2 .

- (e) *Find- α -Program*. Input (m, s) .
 - Run the programs above to get α_1 and α_2 .
 - Compute $\alpha_3 = FC(m, s)$
 - If $\alpha \leftarrow \max\{\alpha_1, \alpha_2, \alpha_3\}$.
 - If $\alpha = \alpha_3$ then output
 - If both runs say that FC and HALF did not work OR the max value you got was from FC then output $f(m, s) \leq \alpha$ by FC.
 - Let α be the max value of $\{\alpha_1, \alpha_2\}$ such that $f(m, s) \leq \alpha$ was verified.
If it was verified by FC then output $f(m, s) \leq \alpha$ by FC.
If it was verified by HALF then output $f(m, s) \leq \alpha$ by HALF with parameters $(V, s_{V-1}, s_V, \beta, \gamma)$.

5. (20 points) Write a program that will on input M, S with $M > S$, do the following:

For $s = 3$ to S

For $m = s + 1$ to M

If m, s are relatively prime

Run Find- α -Program on (m, s) .

The output should be a table like the following (we do the case of $S = 4$ and $M = 7$, though the data is not correct).

m	s	Did it work?	α	Method
3	4	Y	$\frac{5}{12}$	FC
3	5	N		
3	7	Y	$\frac{13}{30}$	$HALF$
4	5	Y	$\frac{7}{24}$	$HALF$
4	7	N		

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6. (15 points) Emily might *teach* 250H in Spring 2023 (Bill is going on sabbatical). She will need help designing problems! In this problem you will help her!

Use constructive strong induction to give TEN 2-tuples (A, B) of RATIONALS with $0 < A, B < 1$ such that the following problem could be asked (that is, what Emily is asking the students to prove is true).

Let a_n be defined as follows.

$$a_0 = 10$$

$$(\forall n \geq 1)[a_n = a_{\lfloor An \rfloor} + a_{\lfloor Bn \rfloor} + n]$$

Show by strong induction that

$$(\forall n \geq 1)[a_n \leq 10n]$$

Include Base Case, IH, and IS.

(Hint: Freely use that $\lfloor An \rfloor \leq An$ and $\lfloor Bn \rfloor \leq Bn$.)

SOLUTION

IB: $a_1 = 10 \leq 10 \times 1$. So this does not impose any condition on (A, B) .

IH: For all $n' < n$, $a_{n'} \leq Cn'$.

IS:

$$a_n = a_{\lfloor An \rfloor} + a_{\lfloor Bn \rfloor} + n \leq 10 \lfloor An \rfloor + 10 \lfloor Bn \rfloor + n \leq 10An + 10Bn + n$$

Want this to be $\leq 10n$. So want

$$10An + 10Bn + n \leq 10n$$

$$10A + 10B + 1 \leq 10$$

$$10(A + B) \leq 9$$

$$A + B \leq \frac{9}{10}.$$

Our ten tuples are

$(0.1, 0, 1), (0.1, 0.2), \dots, (0.1, 0.9)$ and $(0.2, 0.2)$.

END OF SOLUTION

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7. (15 points) For this problem the phrase *10-sided die* means that there are 10 sides, LABELLED $1, \dots, 10$ and each one has probability $\frac{1}{10}$ of being rolled. Similar for *n-sided die* for any n .

Klingon's play the following solitaire dice game:

- The player rolls a 10-sided dice.
- He then decides if he wants to roll a die with EITHER 2 sides, 3 sides, \dots , 10 sides.
- If the total of the two rolled dice is EITHER a square or cube then he WINS. If not then he LOSES.

Worf seeks your help in playing this game.

- (a) (5 points) (You can do this one without a program and probably should to get more of a feel for whats going on.) Worf rolls a 1 on the first die. Find, for each $1 \leq d \leq 10$, the probability that Worf will win if he uses a d -sided die for the second roll. Express both as a fraction and as a decimal to 3 places. Which die should he use to maximize his probability of winning?

The format of your answer should be as follows (the numbers are made up and of course you won't have DOT DOT DOT.)

d	Prob of winning as fraction	Prob of winning as decimal
1	$\frac{1}{3}$	0.333
\vdots	\vdots	\vdots
10	$\frac{1}{13}$	0.077

Worf should use X sided die which gives him prob Y of winning.

- (b) (10 points) (You will want to write a program for this part) For $1 \leq r \leq 10$ find
- $1 \leq d \leq 10$ such that, if Worf roles a r on the first roll, he should choose a d -sided die on the second roll to maximize his probability of winning. If there is more than one d (For example, both rolling a 3-sided die or a 5-sided die give the same probability of winning, and its the highest probability) then give BOTH.

- ii. The probability that Worf wins if he takes your advice. Present both a fraction and a decimal to 3 places.

The data should be in the format below. Note that the numbers are made up and of course you won't have a DOT DOT DOT.

r	Best d 's	Prob of winning as fraction	Prob of winning as decimal
1	8	$\frac{3}{8}$	0.375
2	6	$\frac{1}{3}$	0.333
\vdots	\vdots	\vdots	\vdots
10	3, 4	$\frac{2}{7}$	0.286

SOLUTION

If Worf rolls a 1 then

If he uses a 1 sided die then his prob of winning is 0

If he uses a 2 sided die then his prob of winning is 0

If he uses a 3 sided die then his prob of winning is $\frac{1}{3}$

If he uses a 4 sided die then his prob of winning is $\frac{1}{4}$

If he uses a 5 sided die then his prob of winning is $\frac{1}{5}$

If he uses a 6 sided die then his prob of winning is $\frac{1}{6}$

If he uses a 7 sided die then his prob of winning is $\frac{2}{7}$

If he uses an 8 sided die then his prob of winning is $\frac{3}{8}$

If he uses a 9 sided die then his prob of winning is $\frac{3}{10}$

Worf should use the 8 sided die. Then he has

END OF SOLUTION