1. (0 points but please DO IT) What is your name?

2. (40 points) Given $x, y, z$ a function $a_n(x, y, z)$ be defined by

$$
a_1 = x
$$
$$
a_2 = y
$$
$$
a_3 = z
$$
$$
(\forall n \geq 3)[a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}].
$$

(a) (20 points) Find $\alpha$ (AN APPROXIMATION TO 5 PLACES! THE CLOSED FORM IS UGLY!) such that

$$
(\forall n \geq 1)[a_n \sim \alpha^n]
$$

This $\alpha$ should only depend on the recurrence and not on $x, y, z$. (You will get three values of $\alpha$. Two of them will be complex numbers of length < 1 so as $n$ goes to infinity they are negligible. The third will be a real > 1 and that is what we want.)

(b) (0 points but you will need this) Write a program that will, given $x, y, z, n$, generate $a_1, a_2, \ldots, a_n$

(c) (0 points but you will need this) Write a program that will, given $x, y, z, n$, generate $\alpha^1, \alpha^2, \ldots, \alpha^n$

(d) (0 points but you will need this) Write a program that will, given $x, y, z, n$, run the two programs above and then generate

$$
\frac{\alpha^1}{a_1}, \frac{\alpha^2}{a_2}, \ldots, \frac{\alpha^n}{a_n}.
$$

This sequence should be roughly constant. Call that constant $C(x, y, z)$.
(e) (20 points) For $1 \leq x, y, z \leq 5$ find $C(x, y, z)$. Put it into a table like this:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$C(x, y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

(You will not have the $:$ and you will have the $C(x, y, z)$ column filled in.

(f) (0 points) Which of $x, y, z$ affects $C(x, y, z)$ the most? In what direction?
3. (40 points) Given \( n \) we want to write \( n \) as a sum of cubes of INTEGERS.

Note that

- 23 is the sum of 9 cubes of NATURALS, but NOT 8.
- 23 is the sum of 5 cubes of INTEGERS: 
  \[ 23 = 3^3 + (-1)^3 + 4 \times (-1)^3. \]

Let \( n \in \mathbb{N} \). NCUZ(\( n \)) is the least number such that \( n \) can be written as the sum of NCUZ(\( n \)) cubes of INTEGERS. Clearly NCUZ(\( n \)) \( \leq n \).

In this problem you will write a programs that will, given \( n \in \mathbb{N} \), find \( A[n] \), a bound on NCUZ(\( n \)).

Note the following:

\( n \) can be written as the sum of \( k \) cubes IFF \( -n \) can be written as the sum of \( k \) cubes.

Consider the following thought experiment:

You want to find \( A[23] \). So you look at using

- \( 1^3 \): So then the answer would be \( 1 + A[23 - 1^3] = 1 + A[22] \).
- \( 2^3 \): So then the answer would be \( 1 + A[23 - 2^3] = 1 + A[15] \).
- \( 3^3 \): So then the answer would be \( 1 + A[23 - 3^3] = 1 + A[-4] = 1 + A[4] \).
- \( 4^3 \): So then the answer would be \( 1 + A[23 - 4^3] = 1 + A[-41] = 1 + A[41] \).

CAN’T USE THIS- we do not know \( A[41] \) while doing \( A[23] \).

We can use \( j^3 \) so long as \( |23 - j^3| \leq 22 \).

More generally, while looking at \( A[i] \) can use \( j \) such that \( |i - j^3| \leq i - 1 \).

On the next page we ask the question we want formally.
(a) (10 points) Show that if \(-n\) is the sum of \(k\) integer cubes, then \(n\) is the sum of \(k\) integer cubes.

(b) (0 points but you need to do this for the next part.) Write a program that will, given \(n\), find, for all \(0 \leq i \leq n\), a number \(A[i]\) such that \(i\) can be written as the sum of \(A[i]\) cubes of INTEGERS. Here is how the program will work.

- \(A[0] \leftarrow 0\) (0 can be written as the sum of 0 cubes).
- \(A[1] \leftarrow 1\) (1 can be written as the sum of 1 cube).
- For \(i \leftarrow 2\) to \(n\)

\[ A[i] = 1 + \min\{A[|i - j^3|]: 0 \leq |i - j^3| \leq i - 1\}. \]

(c) (10 points) Run the program on \(n = 10000\), so you get \(A[1], \ldots, A[10000]\).

i. How many numbers took 1 cube? (For how many \(1 \leq i \leq 10000\) is \(A[i] = 1\)?)

ii. How many numbers took 2 cubes? (For how many \(1 \leq i \leq 10000\) is \(A[i] = 2\)?)

iii. How many numbers took 3 cubes? (For how many \(1 \leq i \leq 10000\) is \(A[i] = 3\)?)

iv. ETC- until no numbers required that many cubes.

(d) (20 points) Email your code to Emily. She will run it on many numbers so make sure it is correct.
4. (20 points) On April Fools Day Bill wants to pull the following trick on Emily:

Bill gives Emily a sequence 1, 2, 3, asks her to guess the next number.

She will (quite reasonably) guess 4.

Bill will then produce a polynomial \( p \) such that

\[
\begin{align*}
p(1) &= 1 \\
p(2) &= 2 \\
p(3) &= 3 \\
p(4) &= 1000
\end{align*}
\]

and then say FOOLED YOU! The answer was 1000. (Emily will then roll her eyes.)

Help Bill out! Give a polynomial \( p \) with those values. Try to make the degree as low as possible.

You CAN use tools on the web. If so, tell us what they are.
5. (Extra Credit)

(a) Prove for all \( n \), for all \( a \geq 2^n \), DUP wins \( (L_{2^n}, N + Z + N^*; n) \)

(b) Prove for all \( n \), for all \( a \geq 2^n \), DUP wins \( (L_{2^n}, N + Z + Z + N^*; n) \)