**Homework 9 MORALLY Due Apr 18 at 9:00AM**
**WARNING: THIS HW IS FIVE PAGES LONG!!!!!!!!!!!!!!!!!**

1. (0 points but please DO IT) What is your name?

2. (30 points) Bill has the following:

   - A fair 6-sided die. So the $\Pr(1) = \cdots = \Pr(6) = \frac{1}{6}$.
   - A bias die with
     $\Pr(1) = \Pr(2) = \Pr(3) = \frac{1}{4}$
     $\Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{12}$

   Emily picks one of these die at random (each with prob $\frac{1}{2}$).

   (a) (15 points) If she rolls it $n$ times and gets $n$ 1's, what is the prob she picked the biased die?

   (b) (15 points) If she rolls it $n$ times and gets $n$ 6's what is the prob she picked the biased die?

**SOLUTION**

Let $F$ be picked Fair Die.

Let $B$ be picked Bias Die.

Let $A$ be the event that Emily rolled the die $n$ times and got all 1's.

$$\Pr(B \mid A) = \Pr(A \mid B)\frac{\Pr(B)}{\Pr(A)}$$

We now compute these probabilities separately.

$\Pr(A \mid B) = \frac{1}{4^n}$

$\Pr(B) = \frac{1}{2}$.

$\Pr(A) = \Pr(A \mid B))\Pr(B) + \Pr(A \mid F)\Pr(F)$

$= \frac{1}{4^n} \times \frac{1}{2} + \frac{1}{6^n} \times \frac{1}{2}$

Hence

Lets put it all together

$$\Pr(B \mid A) = \frac{1}{4^n}\frac{1/2}{(1/2)(1/4^n + 1/6^n)} = \frac{1}{4^n}\frac{1}{(1/4^n + 1/6^n)} = \frac{1}{1 + (2/3)^n}$$

**END OF SOLUTION**

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3. (25 points) Emily tosses $m$ balls into $n$ boxes at random. Assume $m \ll n$.

   (a) (15 points) What is the probability that at least FOUR balls are in the same box. (You may use the approximations we used for the problem of THREE balls.)

   **SOLUTION**

   Prob that $i, j, k, l$ in the same box is $\frac{n}{n^4} = \frac{1}{n^3}$.

   Prob that $i, j, k, l$ NOT in the same box is $1 - \frac{1}{n^3} \sim e^{-1/n^3}$.

   Prob that no 4-set is in the same box is approx

$$(e^{-1/m^3})^{\binom{m}{4}} \sim e^{-m^4/24n^3}$$

   Prob that SOME 4-set is in the same box is approx

$$1 - e^{-m^4/24n^3}$$

   **END OF SOLUTION**

   (b) (10 points) Let $n$ be fixed and large. Fill in the following statement:

   *If $m = XXX$ then the prob of having 4 people in a room is OVER $\frac{1}{2}$ and $m$ is close to the least such value of $m$.* (HINT: Use Part 1 of this problem.)

   **SOLUTION**

$$1 - e^{-m^4/24n^3} > \frac{1}{2}$$

$$\frac{1}{2} > e^{-m^4/24n^3}$$

$$-\ln(2) > -\frac{m^4}{24n^3}$$

$$\ln(2) < \frac{m^4}{24n^3}$$

$$24\ln(2)n^3 < m^4$$

$$m > (24\ln(2))^{1/3} n^{3/4}$$

**END OF SOLUTION**

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4. (25 points) Do a COMBINATORIAL PROOF (NOT algebraic, NOT by induction) for the following statement:

   *For all* $n \geq 0$, $\sum_{s=0}^{n} \binom{n}{s} 2^s = 3^n$.

   (HINT: The Right Hand Side is the answer to the question:

   *How many ways can you 3-color* $\{1, \ldots, n\}$.

   Argue that the Left Hand Side solves this same problem.)

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5. (25 points) We are playing with a normal Earth-poker: 13 ranks, 4 suites, hands of size 5.

   What is the probability that a hand has a flush OR a straight but NOT a straight-flush. Give it both in terms of notation like $\binom{52}{5}$ and an actual number like 0.0414 (to 4 places).