Homework 11 Due May 9 at 9:00AM. NO DEAD CAT
Emily will go over the HW in Recitation on Monday May 9

1. (0 points but please DO IT) What is your name?

2. (30 points) In this problem we look at the problem of dividing 8 muffins for 7 people so that everyone gets $\frac{8}{7}$. Recall that $f(8,7)$ is the size of the smallest piece in an optimal protocol.

   (a) (5 points) Use the Floor-Ceiling Formula to get an upper bound on $f(8,7)$. Express as both a fraction and in decimal up to 3 places.

   (b) (15 points) Use the HALF method to show that $f(8,7) \leq \frac{5}{14}$. You can assume that each muffins is cut into 2 pieces so that there are 16 pieces. You can assume that nobody gets just 1 share (if they did then they would have 1 muffins, but they should get $\frac{8}{7} > 1$).

   (c) (10 points) Give a PROTOCOL that achieves the bound $\frac{5}{14}$. We give the format we want for the $f(5,3)$ problem. Do a similar format.

   $f(5,3) \geq \frac{5}{14}$:
   i. Divide 1 muffins ($\frac{6}{12}, \frac{6}{12}$).
   ii. Divide 4 muffins ($\frac{5}{12}, \frac{7}{12}$).
   iii. Give 2 students ($\frac{6}{12}, \frac{7}{12}, \frac{7}{12}$).
   iv. Give 1 students ($\frac{5}{12}, \frac{5}{12}, \frac{5}{12}$).
SOLUTION

a) I’ll do this with a FC-style proof rather than use the formula.
Assume all of the muffins are cut into exactly 2 pieces, so there are 16 pieces.

Since there are 7 student, some student gets $\geq \left\lceil \frac{16}{7} \right\rceil = 3$ pieces. One of those pieces is $\leq \frac{8 \frac{1}{3}}{7} = \frac{8}{21}$.

Since there are 7 student, some student gets $\leq \left\lfloor \frac{16}{7} \right\rfloor = 2$ pieces. One of those pieces is $\geq \frac{8 \frac{1}{2}}{7} = \frac{4}{7}$. Hence that pieces buddy is $\leq 1 - \frac{4}{7} = \frac{3}{7}$.

Hence some piece is $\leq \max\{\frac{8}{21}, \frac{3}{7}\} = \frac{8}{21} \sim 0.38$. 

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b) There are 16 pieces.

*Case 1* Some student has \( \geq 4 \) shares. That student has some share \( \leq \frac{8}{7} \times \frac{1}{4} = \frac{2}{7} < \frac{5}{14} \).

We know that nobody can get 1 share so everyone gets 2 or 3 shares.

*Case 2* Everyone gets 2 or 3 shares. Let \( s_2 \) be the number of students who get 2 shares and \( s_3 \) be the number of students who get 3 shares.

\[
2s_2 + 3s_3 = 16
\]

\[
s_2 + s_3 = 7
\]

\[
s_2 = 5 \text{ and } s_3 = 2
\]

Our thoughts: Gee, the 2-shares need to be fairly big

*Case 2a* Some 2-share is \( \leq \frac{1}{2} \).

Alice has 2 shares \( x, y \) and \( y \leq \frac{1}{2} \).

\[
x + y = \frac{8}{7},
\]

\[
x = \frac{8}{7} - y \geq \frac{8}{7} - \frac{1}{2} = \frac{9}{14}
\]

The BUDDY of \( x \) has to be \( \leq 1 - \frac{9}{14} = \frac{5}{14} \).

*Case 2b* All 2-shares are \( > \frac{1}{2} \). Since 5 people have 2 2-shares, there are 10 2-shares. So there are 10 shares that are \( > \frac{1}{2} \).

But we originally cut 8 muffins into 2 pieces each, so there are at most 8 pieces \( > \frac{1}{2} \). Hence Case 2b cannot happen.
c) Protocol. Thought process: Smallest piece is \( \frac{5}{14} \), so largest piece is \( \frac{9}{14} \). Lets assume the only sizes are \( \frac{5}{14}, \frac{6}{14}, \frac{7}{14}, \frac{8}{14}, \text{ and } \frac{9}{14} \).

Every gets \( \frac{8}{7} = \frac{16}{14} \). So we need ways that two or three of 5,6,7,8,9 can add up to 16.

We do not use the 14 until the final protocol.

5 + 5 + 6 is the ONLY way that 3 shares can add up to 16.

8 + 8, 7 + 9 are the ONLY ways that 2 shares can add up to 16.

Taking a hint from part b we think that there are 5 students who get 2 shares and 2 who get 3 shares.

SO we need shares of sizes 5,5,6;5,5,6, so 4 shares of size 5 and 2 shares of size 6.

Our first cuts will be four (5,9)’s and two (5,8)’s.

We will then give two people (5,5,6). Lets just Do that and see what’s left.

SO we will have four pieces of size 9 and 2 pieces of size 8. The two of size 8 can go to someone. The size 9’s will need some size 7’s.

(a) Divide 4 muffins \( \{ \frac{5}{14}, \frac{9}{14} \} \).
(b) Divide 2 muffins \( \{ \frac{6}{14}, \frac{8}{14} \} \).
(c) Divide 2 muffins \( \{ \frac{7}{14}, \frac{7}{14} \} \).
(d) Give 2 students \( \{ \frac{5}{14}, \frac{5}{14}, \frac{6}{14} \} \).
(e) Give 1 student \( \{ \frac{8}{14}, \frac{8}{14} \} \).
(f) Give 4 students \( \{ \frac{7}{14}, \frac{9}{14} \} \).
3. (30 points) Let $ZAN$ be the set

$$\{a + b\pi : a, b \in \mathbb{Q}\}.$$ 

Let $ZAN[x]$ be the set of polynomials with coefficients in $ZAN$. Is $ZAN[x]$ countable or uncountable? Justify your answer.
4. (30 points) Let \( BILL \) be the set of functions \( f \) such that

(a) The domain is \( \mathbb{N} \)
(b) The co-domain is the primes.
(c) The function is strictly increasing.

Is \( BILL \) countable or uncountable? Justify your answer.
5. (10 points)

(a) (0 points) Listen to *Muffin Math* by Bill Gasarch and Lance Fortnow on YouTube:
https://www.youtube.com/watch?v=4xQFlsK7jKg
or as much of it as you can stand- though its short.

(b) (0 points) Listen to *The Bolzano-Weirstrauss Rap* by The great Steve Sawin
https://www.youtube.com/watch?v=eM3S74kchoM
or as much of it as you can stand. The students in Ramsey largely
did not get to the end. How do I feel about that? I am down with
that, yes I am down with that.
There are two versions of this song on You Tube- they differ only
on graphics. This one has pictures that help with the math.

(c) (0 points) Here is my collection of funny songs (at least I think
they are funny).
One of the categories is math.
Pick three or more songs at random in that category and listen to
them.

(d) (10 points) Are all three better than the Bolzano-Weirstrauss
Rap? (Hint: YES!) For at least one of the songs give me your
thoughts on it. Tell me your favorite math song from my collec-
tion that you listened to.