Homework 11 Due May 9 at 9:00AM. NO DEAD CAT
Emily will go over the HW in Recitation on Monday May 9

1. (0 points but please DO IT) What is your name?

2. (30 points) In this problem we look at the problem of dividing 8 muffins for 7 people so that everyone gets $\frac{8}{7}$. Recall that $f(8,7)$ is the size of the smallest piece in an optimal protocol.

   (a) (5 points) Use the Floor-Ceiling Formula to get an upper bound on $f(8,7)$. Express as both a fraction and in decimal up to 3 places.

   (b) (15 points) Use the HALF method to show that $f(8,7) \le \frac{5}{14}$. You can assume that each muffins is cut into 2 pieces so that there are 16 pieces. You can assume that nobody gets just 1 share (if they did then they would have 1 muffins, but they should get $\frac{8}{7} > 1$).

   (c) (10 points) Give a PROTOCOL that achieves the bound $\frac{5}{14}$. We give the format we want for the $f(5,3)$ problem. Do a similar format.

   $f(5,3) \ge \frac{5}{12}$:

   i. Divide 1 muffins $(\frac{6}{12}, \frac{6}{12})$.
   ii. Divide 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
   iii. Give 2 students $\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$.
   iv. Give 1 students $\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$.

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SOLUTION

a) I'll do this with a FC-style proof rather than use the formula.

Assume all of the muffins are cut into exactly 2 pieces, so there are 16 pieces.

Since there are 7 student, some student gets $\geq \lceil \frac{16}{7} \rceil = 3$ pieces. One of those pieces is $\leq \frac{8}{7}\frac{1}{3} = \frac{8}{21}$.

Since there are 7 student, some student gets $\leq \lfloor \frac{16}{7} \rfloor = 2$ pieces. One of those pieces is $\geq \frac{8}{7}\frac{1}{2} = \frac{4}{7}$. Hence that pieces buddy is $\leq 1 - \frac{4}{7} = \frac{3}{7}$.

Hence some piece is $\leq \max\{\frac{8}{21}, \frac{3}{7}\} = \frac{8}{21} \sim 0.38$.

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b) There are 16 pieces.

Case 1 Some student has ≥ 4 shares. That student has some share $\leq \frac{8}{7} \times \frac{1}{4} = \frac{2}{7} < \frac{5}{14}$.

We know that nobody can get 1 share so everyone gets 2 or 3 shares.

Case 2 Everyone gets 2 or 3 shares. Let s_2 be the number of students who get 2 shares and s_3 be the number of students who get 3 shares.

$2s_2 + 3s_3 = 16$

$s_2 + s_3 = 7$

$s_2 = 5$ and $s_3 = 2$

Our thoughts: Gee, the 2-shares need to be fairly big

Case 2a Some 2-share is $\leq \frac{1}{2}$.

Alice has 2 shares x, y and $y \leq \frac{1}{2}$.

$x + y = \frac{8}{7}$.

$x = \frac{8}{7} - y \geq \frac{8}{7} - \frac{1}{2} = \frac{9}{14}$

The BUDDY of x has to be $\leq 1 - \frac{9}{14} = \frac{5}{14}$.

Case 2b All 2-shares are $> \frac{1}{2}$. Since 5 people have 2 2-shares, there are 10 2-shares. So there are 10 shares that are $> \frac{1}{2}$.

But we originally cut 8 muffins into 2 pieces each, so there are at most 8 pieces $> \frac{1}{2}$. Hence Case 2b cannot happen.

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c) Protocol. Thought process: Smallest piece is $\frac{5}{14}$ so largest piece is $\frac{9}{14}$. Lets assume the only sizes are $\frac{5}{14}$, $\frac{6}{14}$, $\frac{7}{14}$, $\frac{8}{14}$, and $\frac{9}{14}$.

Every gets $\frac{8}{7} = \frac{16}{14}$. So we need ways that two or three of 5,6,7,8,9 can add up to 16.

We do not use the 14 until the final protocol.

$5 + 5 + 6$ is the ONLY way that 3 shares can add up to 16.

$8 + 8$, $7 + 9$ are the ONLY ways that 2 shares can add up to 16.

Taking a hint from part b we think that there are 5 students who get 2 shares and 2 who get 3 shares.

SO we need shares of sizes 5,5,6;5,5,6, so 4 shares of size 5 and 2 shares of size 6.

Our first cuts will be four $(5, 9)$'s and two $(5, 8)$'s.

We will then give two people $(5, 5, 6)$. Lets just Do that and see whats left

SO we will have four pieces of size 9 and 2 pieces of size 8. The two of size 8 can go to someone. The size 9'swill need some size 7's.

(a) Divide 4 muffins $(\frac{5}{14}, \frac{9}{14})$.
(b) Divide 2 muffins $(\frac{6}{14}, \frac{8}{14})$.
(c) Divide 2 muffins $(\frac{7}{14}, \frac{7}{14})$.
(d) Give 2 students $\{\frac{5}{14}, \frac{5}{14}, \frac{6}{14}\}$.
(e) Give 1 students $\{\frac{8}{14}, \frac{8}{14}\}$.
(f) Give 4 students $\{\frac{7}{14}, \frac{9}{14}\}$.

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3. (30 points) Let ZAN be the set

$$\{a + b\pi \colon a, b \in \mathsf{Q}\}.$$

Let $ZAN[x]$ be the set of polynomials with coefficients in ZAN.

Is $ZAN[x]$ countable or uncountable? Justify your answer.

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4. (30 points) Let $BILL$ be the set of functions f such that

   (a) The domain is N

   (b) The co-domain is the primes.

   (c) The function is strictly increasing.

   Is $BILL$ countable or uncountable? Justify your answer.

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5. (10 points)

   (a) (0 points) Listen to *Muffin Math* by Bill Gasarch and Lance Fortnow on You Tube:

   `https://www.youtube.com/watch?v=4xQFlsK7jKg`

   or as much of it as you can stand- though its short.

   (b) (0 points) Listen to *The Bolzano-Weirstrauss Rap* by *The great Steve Sawin*

   `https://www.youtube.com/watch?v=eM3S74kchoM`

   or as much of it as you can stand. The students in Ramsey largely did not get to the end. How do I feel about that? I am down with that, yes I am down with that.

   There are two versions of this song on You Tube- they differ only on graphics. This one has pictures that help with the math.

   (c) (0 points) Here is my collection of funny songs (at least I think they are funny).

   `http://www.cs.umd.edu/~gasarch/FUN//funnysongs.html`

   One of the categories is math.

   Pick three or more songs at random in that category and listen to them.

   (d) (10 points) Are all three better than the Bolzano-Weirstrauss Rap? (Hint: YES!) For at least one of the songs give me your thoughts on it. Tell me your favorite math song from my collection that you listened to.