1. (15 points)

(a) (5 points) What is the coefficient of $x^2y^3z^4$ in

$$(x + y + z)^9$$

(b) (10 points) What is the coefficient of $x^ay^bz^c$ in

$$(x + y + z)^{a+b+c}$$
2. (20 points–5 points each) The Narns play card games with cards that have ranks in the set \{1, 2, \ldots, r\} and suites in the set \{1, \ldots, s\}. In Narn Poker, each player gets \(h\) cards.

We assume that both \(r\) and \(s\) are squares, so \(\sqrt{r}\) and \(\sqrt{s}\) are natural numbers.

(a) A Square Hand is a hand where all of the cards have square rank. What is the probability of getting a Square Hand? (Its okay if they are of the same suite, or not.)

SOLUTION

The number of hands is \(\binom{rs}{h}\).

The number of cards of square rank is \(\sqrt{r} \times s\). Hence the number of hands where all of the cards have square rank is \(\binom{\sqrt{rs}}{h}\).

Hence the answer is

\[
\frac{\binom{\sqrt{rs}}{h}}{\binom{rs}{h}}.
\]

END OF SOLUTION

(b) A Square Flush is a hand where all of the cards have a square rank. and all of the suites are the same. What is the probability of getting a Square Flush?

SOLUTION

Note that in a square flush you can’t have any rank twice. So first pick the \(h\) squares: \(\binom{\sqrt{r}}{h}\). Then pick the suite: \(s\).

so the answer is

\[
\frac{\binom{\sqrt{r}}{h} \cdot s}{\binom{rs}{h}}.
\]

END OF SOLUTION

(c) An Apple is when you get two of the same rank. There are no other restrictions, so for example, if you had 3 of the same rank, that would still be an Apple.

What is the probability of getting an Apple.
SOLUTION
To form a hand that has a pair you pick the rank: \( r \) ways.
Then you pick the suite for those two cards: \( \binom{s}{2} \) ways.
Then you pick the remaining \( h - 2 \) cards out of the remaining \( rs - 2 \) cards.

\[
\frac{r \times \binom{s}{2} \times \binom{rs-2}{h-2}}{\binom{rs}{2}}
\]

END OF SOLUTION
3. (15 points) In this problem we guide you through the birthday paradox with \( m \) balls in \( n \) boxes where we want the prob that at least \( k \) balls go in the same box is \( \geq \frac{1}{2} \). (HINT: Follow the proof for THREE balls in a box and feel free to use the approximations I use there.)

Assume that \( m \) is much less than \( n \). Assume that \( k \) is much less than both \( n, m \).

We put \( m \) balls into \( n \) boxes at random.

(a) Let \( i_1, \ldots, i_k \) be \( k \) balls. What is the prob they are all in the same box?

**SOLUTION**
\[
\frac{n}{n^k} = \frac{1}{n^{k-1}}.
\]

(b) What is the (approx) probability that NO set of \( k \) is in the same box? (Use three approximations here: (a) that the events are independent, and (b) use \( (1 - x) \) is approximately \( e^{-x} \), and (c) \( \binom{m}{k} \sim \frac{m^k}{k!} \).

**SOLUTION**
\[
(1 - \frac{1}{n^{k-1}}) \binom{m}{k} \sim e^{-m^k/k!n^{k-1}}.
\]

(c) Think of \( n, k \) as being fixed but \( m \) as being varying. Approximately how large does \( m \) have to be so that the prob that \( k \) are in the same box is \( \geq \frac{1}{2} \)?

**SOLUTION**
\[
1 - e^{-m^k/k!n^{k-1}} > \frac{1}{2}
\]
\[
e^{-m^k/k!n^{k-1}} < \frac{1}{2}
\]
\[
-\frac{m^k}{k!n^{k-1}} < \ln(1/2) \sim -0.7
\]
\[
0.7 < \frac{m^k}{k!n^{k-1}}
\]
\[
0.7k!n^{k-1} < m^k
\]
\[ m > (0.7k!n^{k-1})^{1/k} \]