Midterm Two, April 20 8:00PM-10:00PM on Zoom
WARNING: THIS MID IS THREE PAGES LONG!!!!!!!!!!!!!!!!!

1. (15 points)

   (a) (5 points) What is the coefficient of $x^2y^3z^4$ in

   $$(x+y+z)^9$$

   (b) (10 points) What is the coefficient of $x^ay^bz^c$ in

   $$(x+y+z)^{a+b+c}$$

GO TO NEXT PAGE

2. (20 points–5 points each) The Narns play card games with a cards that have ranks in the set $\{1, 2, \ldots, r\}$ and suites in the set $\{1, \ldots, s\}$. In Narn Poker, each player gets h cards.

   We assume that both r and s are squares, so $\sqrt{r}$ and $\sqrt{s}$ are natural numbers.

   (a) A *Square Hand* is a hand where all of the cards have square rank. What is the probability of getting a Square Hand?

   (Its okay if they are of the same suite, or not.)

   SOLUTION

   The number of hands is $\binom{rs}{h}$.

   The number of cards of square rank is $\sqrt{r} \times s$. Hence the number of hands where all of the cards have square rank is $\binom{\sqrt{r}s}{h}$.

   Hence the answer is

$$\frac{\binom{\sqrt{r}s}{h}}{\binom{rs}{h}}.$$

   END OF SOLUTION

   (b) A *Square Flush* is a hand where all of the cards have a square rank and all of the suites are the same.

   What is the probability of getting a Square Flush?

   SOLUTION

   Note that in a square flush you can't have any rank twice. So first pick the h squares: $\binom{\sqrt{r}}{h}$. Then pick the suite: s.

   so the answer is

$$\frac{\binom{\sqrt{r}}{h}s}{\binom{rs}{h}}.$$

   END OF SOLUTION

   (c) An *Apple* is when you get two of the same rank. There are no other restrictions, so for example, if you had 3 of the same rank, that would still be an Apple.

   What is the probability of getting an Apple.

SOLUTION

To form a hand that has a pair you pick the rank: r ways.

Then you pick the suite for those two cards: $\binom{s}{2}$ ways.

Then you pick the remaining $h-2$ cards out of the remaining $rs-2$ cards.

$$\frac{r \times \binom{s}{2} \times \binom{rs-2}{h-2}}{\binom{rs}{2}}$$

END OF SOLUTION

GO TO NEXT PAGE

3. (15 points) In this problem we guide you through the birthday paradox with m balls in n boxes where we want the prob that at least k balls go in the same box is $\geq \frac{1}{2}$. (HINT: Follow the proof for THREE balls in a box and feel free to use the approximations I use there.)

   Assume that m is much less than n. Assume that k is much less than both n, m.

   We put m balls into n boxes at random.

   (a) Let $i_1, \ldots, i_k$ be k balls. What is the prob they are all in the same box?

   SOLUTION

   $\frac{n}{n^k} = \frac{1}{n^{k-1}}$.

   (b) What is the (approx) probability that NO set of k is in the same box? (Use three approximations here: (a) that the events are independent, and (b) use $(1-x)$ is approximately e^{-x}, and (c) $\binom{m}{k} \sim \frac{m^k}{k!}$.

   SOLUTION

   $(1 - \frac{1}{n^{k-1}})^{\binom{m}{k}} \sim e^{-m^k/k!n^{k-1}}$.

   (c) Think of n, k as being fixed but m as being varrying. Approximatly how large does m have to be so that the prob that k are in the same box is $\geq \frac{1}{2}$?

   SOLUTION

$$1 - e^{-m^k/k!n^{k-1}} > \frac{1}{2}$$

$$e^{-m^k/k!n^{k-1}} < \frac{1}{2}$$

$$-\frac{m^k}{k!n^{k-1}} < \ln(1/2) \sim -0.7$$

$$0.7 < \frac{m^k}{k!n^{k-1}}$$

$$0.7k!n^{k-1} < m^k$$

$$m > (0.7k!n^{k-1})^{1/k}$$