Untimed Midterm. Morally Due April 11
WARNING: THIS MIDTERM IS FOUR PAGES LONG!!!!!!!!!!!!!!!!!

1. (16 points)

   (a) (0 points) In this problem C is the complex numbers.

   Write a program that does the following: Given A, B consider the recurrence:

   $a_0 = 1$

   $a_1 = 2$

   $(\forall n \geq 2)[a_n = Aa_{n-1} + Ba_{n-2}]$.

   FIND $C, D, \alpha_1, \alpha_2 \in \mathsf{C}$ (use an approximation to 5 places) such that

$$a_n = C\alpha_1^n + D\alpha_2^n.$$

   (b) (0 points but you will need this) Write a program that will, given M, run the program in part a for all $1 \leq A \leq M$ and $-M \leq B \leq M$ and generates a table of the following form:

   $M = 2$:

A	B	α_1	α_2	$\max\{\alpha_1, \alpha_2\}$
1	-2	$1.2 + i$	$2.3 - i$	$2.3 - i$
1	-1	2.2	4.3	4.3
1	0	8.2	1.3	8.2
1	1	9.2	11.3	11.3
1	2	19.2	111.3	111.3
2	-2	1.2	2.3	2.3
2	-1	2.2	4.3	4.3
2	0	8.2	1.3	8.2
2	1	9.2	11.3	11.3
2	2	19.2	111.3	111.3

   (for complex number $a + bi$ the size is $a^2 + b^2$. We use this for defining the max.)

GOTO NEXT PAGE FOR MORE ON THIS PROBLEM

(c) (0 points) Email Emily your code.

(d) (5 points) IF you ran the code on M, how many rows will the program generate? Show your work in deducing the number.

(e) (11 points) Run the code on $M = 3$ and submit the table.

(f) (Extra Credit) Say something intelligent about how A affects MAX ALPHA and how B affects MAX ALPHA. Which has a bigger effect?

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2. (16 points) Emily might *teach* 250H in Spring 2023 (Bill is going on sabbatical). She will need help designing problems! In this problem you will help her!

She wants to ask a question of the following form (With A, B, C replaced by positive natural numbers).

HERE IS THE PROBLEM SHE WANTS TO ASK:

Let a_n be defined as follows.

$a_1 = 5$

$(\forall n \geq 2)[a_n = Ba_{n-1}^2 + Ca_{\lfloor n^{1/3} \rfloor}]$

Show by strong induction that

$(\forall n \geq 1)[a_n \equiv 5 \pmod{12}]$

Include Base Case, IH, and IS.

Now for YOUR PROBLEM: Use constructive induction to find 9 pairs (B, C) such that

$$(\forall n \geq 1)[a_n \equiv 5 \pmod{12}].$$

You will need to have a Base Case, IH, and IS.

SOLUTIONS

All $\equiv$ are mod 12

Base Case $a_1 = 5 \equiv 5$.

IH Assume that, for all $0 \leq i \leq n-1$, $a_i \equiv 5$. Note in particular that $a_{n-1} \equiv 5$ and $a_{\lfloor n^{1/3} \rfloor}] \equiv 5$.

IS

$$a_n = Ba_{n-1}^2 + Ca_{\lfloor n^{1/3} \rfloor} \equiv B \times 5^2 + C \times 5 \equiv 25B + 5C \equiv B + 5C$$

So we need

$$B + 5C \equiv 5$$

$$B \equiv -5C + 5 \equiv 7C + 5$$

We will list all (B, C) that satisfy this.

C	$B = 7C + 5$
1	0
2	7
3	2
4	9
5	4
6	11
7	6
8	1
9	8

END OF SOLUTIONS

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3. (18 points- 6 points each) In this problem all of the x_i are natural numbers. And remember that 0 is a natural number.

   (a) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq 0) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$

   (b) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq 1) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$

   (c) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq 2) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$

   (d) (Extra Credit) How many elements are in the following set:

   $$\{(x_1, \ldots, x_n) \colon (x_i \geq i) \wedge (x_1 + \cdots + x_{10} = 100)\}.$$