The Sequence Mod 4
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1 Introduction

Consider the sequence

$a_1 = 1$

$(\forall n \geq 2)[a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}]$

Empirically we noticed that $(\forall n)[a_n \not\equiv 0 \pmod 4]$. We will prove this.

Henceforth $\equiv$ means $\equiv \pmod 4$.

The first few terms of the sequence mod 4 are

$$1, 2, 3, 1, 3, 2, 1, 2, 3, 1, 3, 1$$

This pattern indicates two things:

- If n is odd then $a_n \equiv 1, 3$.
- If you remove the 2's from the sequence you get

$$1, 3, 1, 3, 1, 3$$

Our main theorem proves both of these and then (easily) that the terms of the sequence is never $\equiv 0$.

Theorem 1.1 *All $\equiv$ are mod 4.*

1. $(\forall n \geq 0)[a_{2n+1} \equiv 1, 3]$.
2. $(\forall n \geq 1)$
   (a) *If $a_n \equiv 1$ then either $a_{n-1} \equiv 3$ or $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 3$.*
   (b) *If $a_n \equiv 3$ then either $a_{n-1} \equiv 1$ or $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 1$.*
3. $a_n \not\equiv 0$.

Proof: The following equations will be used throughout and are easily verified.

EQ1: $a_{2n-1} = a_{2n-2} + a_{n-1}$
EQ2: $a_{2n} = a_{2n-1} + a_n$
EQ3: $a_{2n+1} = a_{2n} + a_n$
EQ4: $a_{2n+1} = a_{2n-1} + a_n$.

1) We prove this by induction on n.

Base Case $n = 0$. $a_1 = 1 \equiv 1$.

IH $a_{2n-1} \equiv 1, 3$.

IS Using EQ4 we have $a_{2n+1} = a_{2n-1} + 2a_n$. Since $a_{2n-1} \equiv 1, 3$, $a_{2n+1} \equiv 1, 3$.

2) We prove this by induction on n. We will assume the theorem for all $m \leq 2n-1$ and prove it for $2n$ and $2n+1$.
Base Case We omit this for now.

IH $(\forall m \leq 2n-1)$ the theorem holds.

IS We first look at a_{2n}.
2a) By Part 1, $a_{2n-1} \not\equiv 0$, hence $a_{2n-1} \equiv 1, 3$. We will do the $a_{2n-1} \equiv 1$ case and leave the $a_{2n-1} \equiv 3$ case to the reader. We have cases based on a_n. By the IH $a_n \equiv 1, 2, 3$.

Case 1 $a_n \equiv 1$.

$$\text{EQ2: } a_{2n} = a_{2n-1} + a_n \equiv 1 + 1 \equiv 2.$$

$$\text{EQ3: } a_{2n+1} = a_{2n} + a_n \equiv 2 + 1 \equiv 3.$$

Case 2 $a_n \equiv 2$.

$$\text{EQ2: } a_{2n} = a_{2n-1} + a_n \equiv 1 + 2 \equiv 3.$$

$$\text{EQ3: } a_{2n+1} = a_{2n} + a_n \equiv 3 + 2 \equiv 1.$$

Case 3 $a_n \equiv 3$. We show this case cannot occur. Since $a_n \equiv 3$, by the IH, either (1) $a_{n-1} \equiv 1$ or (2) $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 1$.

Case 3.1 $a_{n-1} \equiv 1$.

$$\begin{aligned} \text{EQ1: } \quad a_{2n-1} &= a_{2n-2} + a_n \\ 1 &\equiv a_{2n-2} + 1 \\ a_{2n-2} &\equiv 0 \text{ which contradicts the IH.} \end{aligned}$$

Case 3.2 $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 1$.
We recap and extend what we know
We are assuming $a_{2n-1} \equiv 1$.
From EQ1 $a_{2n-1} = a_{2n-2} + a_{n-1}$.
Putting in $a_{2n-1} \equiv 1$ and $a_{n-1} \equiv 2$ we get $a_{2n-2} \equiv 3$.
From the recurrence we have $a_{2n-2} = a_{2n-3} + a_{n-1}$.
Putting in $a_{2n-2} \equiv 3$ and $a_{n-1} \equiv 2$ we have $3 \equiv a_{2n-3} + 2$, so $a_{2n-3} \equiv 1$.
From the recurrence we have $a_{2n-3} = a_{2n-4} + a_{n-2}$.
Putting in $a_{2n-3} \equiv 1$ and $a_{n-2} \equiv 1$ we get $1 = a_{2n-4} + 1$, so $a_{2n-4} \equiv 0$. This contradicts the IH.

2b) The proof is similar to that of Part 2a.

3) We prove this by cases.

If n is odd then, by Part 1, $a_n \not\equiv 0$.

If n is even then $n+1$ is odd. Then by Part 2, $a_n \not\equiv 0$.

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