A Math Problem about Points in the Unit Square
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1 5 Points and 6 Points

The following is a well known application of the pigeonhole principle.

Theorem 1.1 *For all sets of 5 points in the unit square there exists two points that are* $\le \frac{\sqrt{2}}{2}$ *distance apart. This is optimal: there is a set of 5 points such that the min distance is* $\frac{\sqrt{2}}{2}$.

Proof:
Divide the unit square into four quadrants. By the pigeonhole principle there are two points in some quadrant. By the Pythagorean theorem these two points are the following distance apart:

$$\le \sqrt{\frac{1}{2^2}+\frac{1}{2^2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \sim 0.7071.$$

To achieve this put four points in the four corners and one in the center.
∎

What about 6 points?

Theorem 1.2 *For all sets of 6 points in the unit square there exists two points that are* $\le \frac{\sqrt{2257}}{72} \sim 0.65983$ *apart.*

Proof: Break the unit square into 5 rectangles as follows:
Draw a vertical line that divides the rectangle into two rectangles:
One is $x \times 1$.
One is $(1-x) \times 1$.
Divide the $x \times 1$ rectangle into two equal pieces, so you have two $x \times \frac{1}{2}$ rectangles. Note that the diagonal of those rectangles is $\sqrt{x^2 + \frac{1}{4}}$.

Divide the $(1-x) \times 1$ rectangle into three equal pieces, so you have three $(1-x) \times \frac{1}{3}$ rectangles. Note that the diagonal of those rectangles is $\sqrt{(1-x)^2 + \frac{1}{9}}$.

We plan to put the 6 points into a unit square so two of them have to be in the same rectangle. Hence we want to equate the two diagonals.

$$x^2 + \frac{1}{4} = (1-x)^2 + \frac{1}{9}.$$

$$\frac{1}{4} = 1 - 2x + \frac{1}{9}$$

$$2x = 1 + \frac{1}{9} - \frac{1}{4} = \frac{31}{36}$$

$$x = \frac{31}{72}$$

With this value of x we have that if there are 6 points there must be 2 that are $\leq$ the following distance apart:

$$\leq \sqrt{x^2 + \frac{1}{4}} = \sqrt{\frac{2257}{5184}} = \frac{\sqrt{2257}}{72} \sim 0.65983.$$

$\blacksquare$

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2 More Generally

Theorem 2.1 *Let $n \in \mathsf{N}$. Let $A, B, C, D \in \mathsf{N}$ and $x \in \mathsf{R}$ such that*

- $AB + CD = n$
- $A = C$
- $x = \frac{1}{2} + \frac{A^2}{2D^2} - \frac{C^2}{2B^2}$
- $0 \le x \le 1$

For all sets of $n+1$ points in the unit square there exists two points that are $\le \sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$ apart.

Proof: We do the proof with parameters A, B, C, D, x but later see that they must satisfy the conditions in the theorem.

Draw a vertical line that divides the rectangle into two rectangles:

One is $x \times 1$.

One is $(1-x) \times 1$.

Divide the $x \times 1$ rectangle on the x side into A equal pieces, so you have A rectangles that are $\frac{x}{A} \times 1$. Divide each of those rectangles on the 1-side into B rectangles that are $\frac{x}{A} \times \frac{1}{B}$ Note that the diagonal of those rectangles is $\sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$.

Divide the $(1-x) \times 1$ rectangle on the x side into C equal pieces, so you have C rectangles that are $\frac{1-x}{C} \times 1$. Divide each of those rectangles on the 1-side into D rectangles that are $\frac{1-x}{C} \times \frac{1}{D}$ Note that the diagonal of those rectangles is $\sqrt{\frac{(1-x)^2}{C^2} + \frac{1}{D^2}}$.

The number of rectangles is $AB + CD$ so we need $AB + CD = n$.

We want the diagonals to be the same so we want

$$\frac{x^2}{A^2} + \frac{1}{B^2} = \frac{(1-x)^2}{C^2} + \frac{1}{D^2}.$$

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We want the diagonals to be the same so we want

$$\frac{x^2}{A^2} + \frac{1}{B^2} = \frac{(1-x)^2}{C^2} + \frac{1}{D^2}.$$

Lets make $A = C$ so the algebra works out better.

$$\frac{x^2}{A^2} + \frac{1}{B^2} = \frac{(1-x)^2}{A^2} + \frac{1}{D^2}.$$

$$\frac{1}{B^2} = \frac{1-2x}{A^2} + \frac{1}{D^2}.$$

$$\frac{2x}{A^2} = \frac{1}{A^2} + \frac{1}{D^2} - \frac{1}{B^2}.$$

$$2x = 1 + \frac{A^2}{D^2} - \frac{A^2}{B^2}.$$

$$x = \frac{1}{2} + \frac{A^2}{2D^2} - \frac{A^2}{2B^2} = \frac{1}{2} + \frac{A^2}{2D^2} - \frac{C^2}{2B^2}.$$

Hence there are two points that are $\sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$ $\blacksquare$

We believe that the distance is minimized when A, B, C, D are all close to each other, so all close to $\sqrt{\frac{n}{2}}$.

So here is the program I need written:

1. Input n
2. $X \leftarrow \lceil \sqrt{\frac{n}{2}} \rceil$.
3. $XX \leftarrow \{X-1, X, X+1\}$.
4. For all $(A, B, C, D) \in XX \times XX \times XX \times XX$
   (a) $x \leftarrow \frac{1}{2} + \frac{A^2}{2D^2} - \frac{C^2}{2B^2}$.
   (b) $d \leftarrow \sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$.
   (c) If $AB + CD = n$ and $A = C$ and $0 \le x \le 1$ then output (A, B, C, D, d).

There are so few (A, B, C, D) that we can see which one gives the lowest d and perhaps spot a pattern for which A, B, C, D to use in general.

3 What if we don't take $A = C$?

Theorem 3.1 *Let $n \in \mathsf{N}$. Let $A, B, C, D \in \mathsf{N}$ and $x \in \mathsf{R}$ such that*

- $AB + CD = n$
- *There is a unique x that is both (a) between 0 and 1, and (b) is a root of*
$$\left(\frac{1}{A^2} - \frac{1}{C^2}\right)x^2 + \frac{2}{C^2}x - \frac{1}{C^2} - \frac{1}{D^2}.$$
- $0 \le x \le 1$

For all sets of $n+1$ points in the unit square there exists two points that are $\le \sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$ apart.

Proof: We do the proof with parameters A, B, C, D, x but later see that they must satisfy the conditions in the theorem.

Draw a vertical line that divides the rectangle into two rectangles:

One is $x \times 1$.

One is $(1-x) \times 1$.

Divide the $x \times 1$ rectangle on the x side into A equal pieces, so you have A rectangles that are $\frac{x}{A} \times 1$. Divide each of those rectangles on the 1-side into B rectangles that are $\frac{x}{A} \times \frac{1}{B}$ Note that the diagonal of those rectangles is $\sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$.

Divide the $(1-x) \times 1$ rectangle on the x side into C equal pieces, so you have C rectangles that are $\frac{1-x}{C} \times 1$. Divide each of those rectangles on the 1-side into D rectangles that are $\frac{1-x}{C} \times \frac{1}{D}$ Note that the diagonal of those rectangles is $\sqrt{\frac{(1-x)^2}{C^2} + \frac{1}{D^2}}$.

The number of rectangles is $AB + CD$ so we need $AB + CD = n$.

We want the diagonals to be the same so we want

$$\frac{x^2}{A^2} + \frac{1}{B^2} = \frac{(1-x)^2}{C^2} + \frac{1}{D^2}.$$

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We want the diagonals to be the same so we want

$$\frac{x^2}{A^2} + \frac{1}{B^2} = \frac{(1-x)^2}{C^2} + \frac{1}{D^2}.$$

$$\frac{x^2}{A^2} + \frac{1}{B^2} = \frac{1}{C^2} - \frac{2x}{C^2} + \frac{x^2}{C^2} + \frac{1}{D^2}.$$

$$\left(\frac{1}{A^2} - \frac{1}{C^2}\right)x^2 + \frac{2}{C^2}x - \frac{1}{C^2} - \frac{1}{D^2}.$$

This quadratic equation has 2 roots. Let x be the one that is between 0 and 1 If neither are then this choice of (A, B, C, D) does not work. If both are then let me then flag that case for later study.

We now know that the two points that are $\le \sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$ apart. ∎

We believe that the distance is minimized when A, B, C, D are all close to each other, so all close to $\sqrt{\frac{n}{2}}$. We are not sure how close so we introduce another variable off for the offset.

So here is the program I need written:

1. Input n, off.
2. $X \leftarrow \lceil \sqrt{\frac{n}{2}} \rceil$.
3. $XX \leftarrow \{X - off, \ldots, X + off\}$.
4. For all $(A, B, C, D) \in XX \times XX \times XX \times XX$
   (a) If $AB + CD \neq n$ then go to the next (A, B, C, D).
   (b) Find the roots of
   $$\left(\frac{1}{A^2} - \frac{1}{C^2}\right)x^2 + \frac{2}{C^2}x - \frac{1}{C^2} - \frac{1}{D^2}.$$
   If there is only one that is between 0 and 1 then let x be that root.
   (c) $d \leftarrow \sqrt{\frac{x^2}{A^2} + \frac{1}{B^2}}$.
   (d) Output (n, A, B, C, D, d).