The Muffin Problem

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How it Began

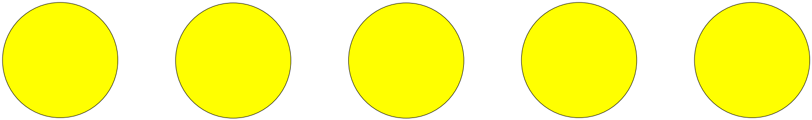
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

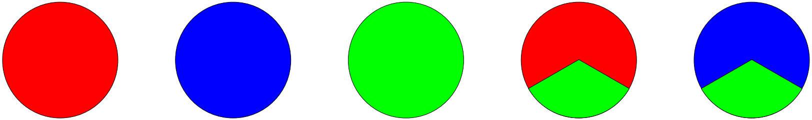
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

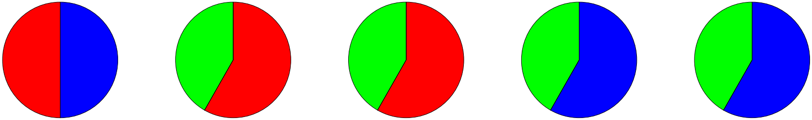
The smallest piece in the above solution is $\frac{1}{3}$.
Is there a procedure with a larger smallest piece?
Work on it with your neighbor

Five Muffins, Three People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Carol	GREEN	$\frac{5}{12}+\frac{5}{12}+\frac{5}{12}+\frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

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5 Muffins, 3 People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

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(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.

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(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students:
Someone gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

What Happened Next?

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Yada Yada Yada- in 2020:

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General Problem

$f(m,s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3) = \frac{5}{12}$ here.

We have two proofs that shown $f(m,s)$ exists, is rational, and is computable.
One use Linear Programming.
One use Integer Programming.

Amazing Results!/Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant !

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Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

What if $m < s$?

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Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

What if $m < s$?

Duality Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.
Hence we will just look at $m > s$.

Floor-Ceiling Thm Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le \mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

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Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

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Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{\mathbf{m}}{\mathbf{s}\lceil \mathbf{2m/s} \rceil}$.

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Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

FC Gives Upper Bound

Give m, s:

$$\mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}$$

Gives an upper bound. So we know

$$(\forall m,s)[f(m,s) \le \mathrm{FC}(m,s)].$$

Is the following true?

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No: If so my book would be about 20 pages.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1.$

$f(3k+1,3) = \frac{3k-1}{6k}, \; k \ge 1.$

$f(3k+2,3) = \frac{3k+2}{6k+6}.$

Note: A Mod 3 Pattern.
Theorem: For all $m \ge 3$, $f(m,3) = \mathrm{FC}(m,3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$f(4k, 4) = 1$ (easy)

$f(1, 4) = \frac{1}{4}$ (easy)

$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = \mathrm{FC}(m, 4)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k,5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1,5) = \frac{5k+1}{10k+5}$. $f(11,5)$?

For $k \geq 2$, $f(5k+2,5) = \frac{5k-2}{10k}$. $f(7,5) = \mathrm{FC}(7,5) = \frac{1}{3}$

For $k \geq 1$, $f(5k+3,5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k+4,5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ **except m=11**, $f(m,5) = \mathrm{FC}(m,5)$.

What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11,5) \ge \frac{13}{30}$.
2. $f(11,5) \le \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5\rceil}, 1 - \frac{11}{5\lfloor 22/5\rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

Options:

1. $f(11,5) = \frac{11}{25}$. Need to find procedure.
2. $f(11,5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11,5)$ in between. Need to find both.
4. $f(11,5)$ unknown to science!

Vote

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Vote WE SHOW $f(11,5) = \frac{13}{30}$. **Exciting** new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M.
The *other piece* from muffin M is the **buddy of x**.

Note that the **buddy** of x is of size

$$1 - x.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

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(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

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That pieces **buddy** is of size:

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

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(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

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$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

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$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets w, x, y, z and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

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Let x be the largest of x, y, z

$$x \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

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GREAT! This is where $\frac{13}{30}$ comes from!

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

HALF Method

The above reasoning can be used to *verify* that $f(11,5) \le \frac{13}{30}$ but could not *generate* the upper bound $\frac{13}{30}$.

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$$(\forall m,s)[f(m,s) \le \min\{\mathrm{FC}(m,s), \mathrm{HALF}(m,s)\}]$$

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For $f(24, 11)$ it fails!

$f(24,11) \le \frac{19}{44}$

Assume $(24,11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\le \frac{19}{44}$.

$f(24,11) \le \frac{19}{44}$

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Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.
5-students: a student who gets 5 shares. s_5 is the number of them.

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$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.
$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

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Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.
Proof: Assume that Alice 5 pieces A, B, C, D, E and $E \geq \frac{20}{44}$.
Since $A+B+C+D+E = \frac{24}{11}$ and $E > \frac{20}{44}$

$$A+B+C+D < \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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Henceforth we assume all 5-shares are in

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Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume that Alice 4 pieces A, B, C, D and $D \le \frac{21}{44}$.

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$$A+B+C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

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The **buddy** of A is of size

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Case 3.5: All Shares in Their Proper Intervals

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Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

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More Refined Picture of What is Going On

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The following picture captures what we know so far.

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{23}{44}$		$\frac{24}{44}$		$\frac{25}{44}$

S4= Small 4-shares
L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|L4|=20$.

$$\underset{\frac{19}{44}}{(}\quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[}\quad 0 \quad \underset{\frac{21}{44}}{](}\quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[}\quad 0 \quad \underset{\frac{24}{44}}{](}\quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

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Claim 2: Every 4-student has at least 3 L4 shares.

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Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

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Contradiction: Each 4-student gets ≥ 3 L4 shares. There are $s_4 = 7$ 4-students. Hence there are ≥ 21 L4-shares. But there are only 20.

More Techniques

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We developed more:
MID, GAP,
EBM (Easy Buddy Match), HBM (Hard Buddy Match),
TRAIN

Upshot

Let

$$A = \{(m, s) \mid 2 \leq s \leq 100 \text{ and } s < m \leq 110 \text{ and } m, s \text{ rel prime}\}$$

There are 3520 pairs (m, s) in A. We solved **all** of them!

- For 2301 of them $f(m, s) = \mathrm{FC}(m, s)$. That is $\sim 65.37\%$.
- For 329 of them $f(m, s) = \mathrm{HALF}(m, s)$. That is $\sim 9.35\%$.
- For 186 of them $f(m, s) = \mathrm{INT}(m, s)$. That is $\sim 5.28\%$.
- For 111 of them $f(m, s) = \mathrm{MID}(m, s)$. That is $\sim 3.15\%$.
- For 240 of them $f(m, s) = \mathrm{EBM}(m, s)$. That is $\sim 6.28\%$.
- For 89 of them $f(m, s) = \mathrm{HBM}(m, s)$. That is $\sim 2.53\%$.
- For 250 of them $f(m, s) = \mathrm{GAP}(m, s)$. That is $\sim 7.10\%$.
- For 13 of them $f(m, s) = \mathrm{TRAIN}(m, s)$. That is $\sim 0.40\%$

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Is the following true: For all m, s, $f(m, s)$ is the min of

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No. Did not work on

- $f(205, 178)$
- $f(226, 135)$
- $f(233, 141)$

The Scott Huddleston Technique

Scott Huddleston has an algorithm that is REALLY FAST and seems to ALWAYS WORK. Erik and Jacob understand it, nobody else does. They have replicated his results and think that YES it solves ALL problems.

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Richard Chatwin independently came up with the same algorithm and proved it worked, but never coded it up. He tells me its poly time and I believe this can be proved, though its not in his paper. His paper is here: `https://arxiv.org/abs/1907.08726`

Interesting VS Important

A Math problem can be

- Interesting or NOT Interesting.
- Important or NOT Important.

These are **not** well defined terms but

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Important Either useful OR about fundamental math concepts.

There are four possibilities.

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4. Not Interesting and Not Important: So far $R(5)$ has not lead to any math of interest and is also not important to find. Same for most Ramsey-type Numbers.

Lessons Learned

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You never know where the next big project will come from!