Small Ramsey Numbers

Exposition by William Gasarch

April 15, 2022
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We state this in terms of colorings of edges of graphs.

For all 2-coloring of the edges of $K_6$ there is a mono $K_3$. 
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We state this in terms of colorings of edges of graphs.

*For all 2-coloring of the edges of $K_6$ there is a mono $K_3$.*

**Question** What if we color the edges of $K_5$?
Coloring of $K_5$ with no Mono $K_3$

This graph is not arbitrary.

$SQ_5 = \{x^2 \pmod{5} : 0 \leq x \leq 4\} = \{0, 1, 4\}$.

- If $i - j \in SQ_5$ then RED.
- If $i - j \notin SQ_5$ then BLUE.
Asymmetric Ramsey Numbers

**Definition** $R(a, b)$ is least $n$ such that for all 2-colorings of $K_n$ there is either a red $K_a$ or a blue $K_b$.

1. $R(a, b) = R(b, a)$.

2. $R(2, b) = b$

3. $R(a, 2) = a$
Theorem \( R(a, b) \leq R(a - 1, b) + R(a, b - 1) \)

Proof

Let \( n = R(a - 1, b) + R(a, b - 1) \). COL: \( ([n]_2) \rightarrow [2] \).

Case 1 \( (\exists v)[\deg_R(v) \geq R(a - 1, b)] \). Look at the \( R(a - 1, b) \) vertices that are RED to \( v \). By Definition of \( R(a - 1, b) \) either

- There is a RED \( K_{a-1} \). Combine with \( v \) to get RED \( K_a \).
- There is a BLUE \( K_b \).
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Case 2 \((\exists v)[\text{deg}_B(v) \geq R(a, b - 1)]\). Similar to Case 1.
\[ R(a, b) \leq R(a - 1, b) + R(a, b - 1) \]

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**Case 2** \((\exists v)[\deg_B(v) \geq R(a, b - 1)]\). Similar to Case 1.

**Case 3**

\((\forall v)[\deg_R(v) \leq R(a - 1, b) - 1 \land \deg_B(v) \leq R(a, b - 1) - 1]\)
\((\forall v)[\deg(v) \leq R(a - 1, b) + R(a, b - 1) - 2 = n - 2]\)

Not possible since every vertex of \( K_n \) has degree \( n - 1 \).
Let's Compute Bounds on $R(a, b)$

- $R(3, 3) \leq R(2, 3) + R(3, 2) \leq 3 + 3 = 6$
- $R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 = 10$
- $R(3, 5) \leq R(2, 5) + R(3, 4) \leq 5 + 10 = 15$
- $R(3, 6) \leq R(2, 6) + R(3, 5) \leq 6 + 15 = 21$
- $R(3, 7) \leq R(2, 7) + R(3, 6) \leq 7 + 21 = 28$

Can we make some improvements to this?

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Can we make some improvements to this? YES!
\( R(3, 4) \leq 9 \)

**Theorem** \( R(3, 4) \leq 9 \).

Let \( COL \) be a 2-coloring of the edges of \( K_9 \).

**Case 1** \((\exists v)[\deg_R(v) \geq 4]\). \( v_1, v_2, v_3, v_4 \) are RED to \( v \).

**Case 2** \((\exists v)[\deg_B(v) \geq 6]\). \( v_1, v_2, v_3, v_4, v_5, v_6 \) are BLUE to \( v \).

Either:

1. a RED \( K_3 \), or
2. a BLUE \( K_3 \), which together with \( v \) is a BLUE \( K_4 \).

**NOTE** Can't have any \( \deg_R(v) \leq 2 \).

**Case 3** \((\forall v)[\deg_R(v) = 3]\). The RED subgraph has 9 nodes each of degree 3. Impossible!
Theorem $R(3, 4) \leq 9$.  
Let $\text{COL}$ be a 2-coloring of the edges of $K_9$.  

**Case 1** ($\exists v)[\deg_R(v) \geq 4]$. $v_1, v_2, v_3, v_4$ are RED to $v$.  
If any of $v_i, v_j$ is RED, then $v, v_i, v_j$ are RED $K_3$.  

**Case 2** ($\exists v)[\deg_B(v) \geq 6]$. $v_1, v_2, v_3, v_4, v_5, v_6$ are BLUE to $v$.  
Either:  
(1) a RED $K_3$, or  
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**Lemma** Let $G = (V, E)$ be a graph.

$$V_{even} = \{ v : \deg(v) \equiv 0 \pmod{2} \}$$
$$V_{odd} = \{ v : \deg(v) \equiv 1 \pmod{2} \}$$

Then $|V_{odd}| \equiv 0 \pmod{2}$. 
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Recall that for any graph $G = (V, E)$:

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod{2}.$$
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\[ \sum_{v \in V_{\text{odd}}} \deg(v) \equiv 0 \pmod{2}. \]

Sum of odds $\equiv 0 \pmod{2}$. Must have even numb of them. So $|V_{\text{odd}}| \equiv 0 \pmod{2}$. 
A Generalization of this Trick

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**Key:** $R(2, 4)$ and $R(3, 3)$ were both even!

**Theorem** $R(a, b)$ ≤

1. $R(a, b - 1) + R(a - 1, b)$ always.
2. $R(a, b - 1) + R(a - 1, b) - 1$ if $R(a, b - 1) \equiv R(a - 1, b) \equiv 0 \pmod{2}$
Some Better Upper Bounds

\[
\begin{align*}
R(3, 3) &\leq R(2, 3) + R(3, 2) \leq 3 + 3 = 6. \\
R(3, 4) &\leq R(2, 4) + R(3, 3) \leq 4 + 6 - 1 = 9. \\
R(3, 5) &\leq R(2, 5) + R(3, 4) \leq 5 + 9 = 14. \\
R(3, 6) &\leq R(2, 6) + R(3, 5) \leq 6 + 14 - 1 = 19. \\
R(3, 7) &\leq R(2, 7) + R(3, 6) \leq 7 + 19 = 26 \\
R(4, 4) &\leq R(3, 4) + R(4, 3) \leq 9 + 9 = 18. \\
R(4, 5) &\leq R(3, 5) + R(4, 4) \leq 14 + 18 - 1 = 31. \\
R(5, 5) &\leq R(4, 5) + R(5, 4) = 62.
\end{align*}
\]

Are these tight?
$R(3, 3) \geq 6$

$R(3, 3) \geq 6$: Need coloring of $K_5$ w/o mono $K_3$. 

Note $-1 = 2^2 \mod 5$. Hence $a-b \in \mathbb{S}_2$ iff $b-a \in \mathbb{S}_2$. So the coloring is well defined.
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Vertices are $\{0, 1, 2, 3, 4\}$. 

$COL(a, b) = \text{RED}$ if $a - b \equiv SQ \pmod{5}$, $\text{BLUE} \text{OW}$.

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- Squares mod 5: 1,4.
- If there is a RED triangle then $a - b, b - c, c - a$ all SQ’s. SUM is 0. So

  $$x^2 + y^2 + z^2 \equiv 0 \pmod{5}$$

  Can show impossible

- If there is a BLUE triangle then $a - b, b - c, c - a$ all non-SQ’s. Product of nonsq’s is a sq. So

  $$2(a - b), 2(b - c), 2(c - a)$$

  all squares. SUM to zero- same proof.

**UPSHOT** $R(3, 3) = 6$ and the coloring used math of interest!
$R(4, 4) = 18$

$R(4, 4) \geq 18$: Need coloring of $K_{17}$ w/o mono $K_4$. 

Vertices are \{0, \ldots, 16\}.

Use $\text{COL}(a, b) = \text{RED}$ if $a - b \equiv \text{SQ} \pmod{17}$, BLUE OW.

Same idea as above for $K_5$, but more cases.

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$R(3, 5) = 14$

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Vertices are \( \{0, \ldots, 13\} \).

Use
\[ \text{COL}(a, b) = \text{RED if } a - b \equiv \text{CUBE} \pmod{14}, \text{BLUE OW.} \]
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**UPSHOT** $R(3, 5) = 14$ and the coloring used math of interest!
$R(3, 4) = 9$

This is a subgraph of the $R(3, 5)$ graph
$R(3, 4) = 9$

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**UPSHOT** $R(3, 4) = 9$ and the coloring used math of interest!
Can we extend these Patterns?

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No other \( R(a, b) \) are known using NICE methods.
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\( R(5, 5) \)– I will give you a paper to read on that soon.
Revisit those Numbers


- $R(3,3) \leq 6$. TIGHT. Int
- $R(3,4) \leq 9$. TIGHT. Int
- $R(3,5) \leq 14$. TIGHT. Int
- $R(3,6) \leq 19$. KNOWN: 18. Upper Bd Bor, Lower Bd Int
- $R(3,7) \leq 26$. KNOWN: 23. Upper Bd Bor, Lower Bd Int
- $R(4,4) \leq 18$. TIGHT. Int
- $R(4,5) \leq 31$. KNOWN: 25. Both bd Bor
- $R(5,5) \leq 62$. KNOWN: Will see it in the paper I give out.
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Moral of the Story

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1. At first there seemed to be interesting mathematics with mods and primes leading to nice graphs. (Joel Spencer) *The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.*

2. Seemed like a nice Math problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.