Sets of Functions that are Uncountable

Exposition by William Gasarch

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The set of all Functions from $\mathbb{N}$ to $\mathbb{N}$

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For all i, F cannot be f_i since $F(i) \neq f_i(i)$.
So F is NOT on the list and IS from $\mathbb{N}$ to $\mathbb{N}$, contradiction.

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For future proofs you must check both properties.

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$$F(x) = f_x(x) + 2.$$

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$$F(x) = (f_x(x) + 1)^2$$

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$(f_x(x) + 1)^2 = f_x(x)^2 + 2f_x(x) + 1$
IF $f_x(x)^2 + 2f_x(x) + 1 = f_x(x)$ then $f_x(x)^2 + f_x(x) + 1 = 0$
Only has complex solutions, so can't happen.

The set of all Constant Functions With Domain $\mathbb{N}$

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1) F is NOT on the list. Good!

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1) F is NOT on the list. Good!
2) But F is not constant. So proof fails.

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- If every element of A can be represented with a finite number of bits then A is countable
- If an infinite number of elements of A require an infinite number of bits to be represented, then A is NOT countable.