START

RECORDING

Strong Induction

CMSC 250

Recall The (Weak) Induction Principle

- Recall that it works since, to prove $P(17)$ we have

- $P(0)$
- $P(0) --> P(1)$
- $P(1) --> P(2)$
- $\vdots$
- $P(16) --> P(17)$

- Note: We have a lot of information we are not using!
- We have $P(1), P(2), \ldots, P(16)$.

Strong Induction: The Principle

- BS: $P(0)$
- IS: $(\forall\, n \geq\ 1)[P(0) \wedge \cdots\ P(n-1)\ \Rightarrow P(n)]$
- From these two you can deduce
  - For all $n, P(n)$.

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- From these two you can deduce
  - For all $n, P(n)$.
- Why does this work. To prove $P(17)$ we have
  - $P(0)$
  - $P(0) \Rightarrow P(1)$. So we have $P(1)$
  - $P(0) \wedge\ P(1)\ \ \Rightarrow P(2)$. So we have $P(2)$
  - $P(0) \wedge\ P(1) \wedge\ P(2) \Rightarrow P(3)$. So we have $P(3)$.
  - $\vdots$
  - $P(0) \wedge \cdots \wedge P(16)\ \ \Rightarrow P(17)$. So we have $P(17)$.

Utility of Strong Induction

- Enormous
  - Correctness of algorithms
  - Growth of structures like trees, graphs, lists, strings, sets

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  - How many ways have we talked about that can be used to describe a sequence?

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  - Correctness of algorithms
  - Growth of structures like trees, graphs, lists, strings, sets
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  - How many ways have we talked about that can be used to describe a sequence?

1 2 3 4

1. Outlining terms
2. Recursive definition
3. Closed-form formula

- Also useful in the study of **algorithm correctness`.**

A First Example

- Let a be a sequence such that:

$$a_n = \begin{cases} 1, & n = 0 \\ 8, & n = 1 \\ a_{n-1} + 2 \cdot a_{n-2}, & n \geq 2 \end{cases}$$

- Prove that $a_n = 3 \cdot 2^n + 2(-1)^{n+1}\ , n \in \mathbb{N}$

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- How many elements in my inductive base?

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Inductive Base

$$a_n = \begin{cases} 1, & n = 0 \\ 8, & n = 1 \\ a_{n-1} + 2 \cdot a_{n-2}, & n \geq 2 \end{cases}$$

$$P(n) \Leftrightarrow a_n = 3 \cdot 2^n + 2(-1)^{n+1}$$

- For $n = 0$, $a_0 = 1$ by the definition of a. $P(0)$ says: $a_0 = 3 \cdot 2^0 + 2(-1)^1 = 3 - 2 = 1$.
  So $P(0)$ holds.
- For n = 1, $a_1 = 8$ by the definition of a. $P(1)$ says: $a_1 = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8$.

  So P(1) holds.

Inductive Base

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  So P(1) holds.

Inductive Base established!

Inductive Hypothesis

$P(\boxed{0})$ $P(\boxed{1})$

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$P(n) \Leftrightarrow a_n = 3 \cdot 2^n + 2(-1)^{n+1}$

- Suppose $n \geq 1$. Then, $\forall i \in \{0, 1, \ \ldots, n\}$ assume $P(i)$, i.e

$$a_i = 3 \cdot 2^i + 2(-1)^{i+1}, i = 0, 1, \ldots, n$$

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- Suppose $n = k \geq 1$. Then, $\forall i \in \{0, 1, \ \ldots, n\}$ assume $P(i)$, i.e

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Inductive Hypothesis made!

Inductive Step

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$P(i)\ \forall i = 0,1,\ldots n$

- We will now **<u>prove</u>** $P(n+1)$, i.e

$$a_{n+1} = 3 \cdot 2^{n+1} + 2(-1)^{n+2}$$

Inductive Step

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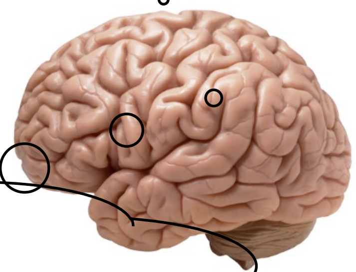

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<u>My goal: Prove</u> $P(n+1)$: $a_{n+1} = 3 \cdot 2^{n+1} + 2(-1)^{n+2}$

- Since $\boldsymbol{n \geq 1 \Rightarrow (n+1) \geq 2}$, we can apply the recursive rule of the sequence.
- From the recursive definition of a_n, we obtain:

$$a_{n+1} = a_n + 2 \cdot a_{n-1} \overset{I.H}{=} 3 \cdot 2^n + 2(-1)^{n+1} + 2 \cdot \left(3 \cdot 2^{n-1} + 2\,(-1)^n\right) =$$

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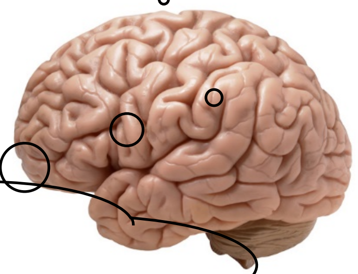

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Inductive step proven!

$$a_{n+1} = a_n + 2 \cdot a_{n-1} \overset{I.H}{=} 3 \cdot 2^n + 2(-1)^{n+1} + 2 \cdot \left(3 \cdot 2^{n-1} + 2\,(-1)^n\right) =$$
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$\square$

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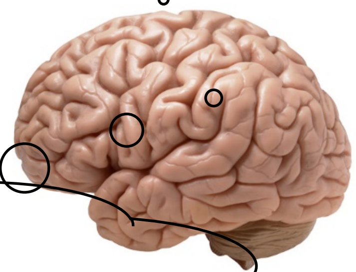

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Proof done!

$\square$

Here's Another

- Suppose that the sequence a_n is as follows:

$$a_n = \begin{cases} 12, & n = 0 \\ 29, & n = 1 \\ 5a_{n-1} - 6a_{n-2}, & n \geq 2 \end{cases}$$

- Then, prove that $a_n = 5 \cdot 3^n + 7 \cdot 2^n, \forall n \in \mathbb{N}$

Inductive Base

- Let the statement to be proven be called $P(n)$. We proceed via strong induction on n.
- **Inductive base:** We want to ***prove*** $P(0), P(1)$.
  - For $n = 0, P(0)$ is $s_0 = 5 \cdot 3^0 + 7 \cdot 2^0 \Leftrightarrow 12 = 12$
  - For $n = 1, P(1)$ is $s_1 = 5 \cdot 3^1 + 7 \cdot 2^1 \Leftrightarrow 29 = 15 + 14$

So the inductive base has been established!

Inductive Hypothesis

- **Inductive Hypothesis:** Let $n \geq 1$. Then, we ***assume*** that, **for all** $i = 0,1, \ldots, n, P(i)$ holds, i.e

$$a_i = 5 \cdot 3^i + 7 \cdot 2^i, \quad i = 0,1, \ldots, n$$

Inductive Step

- **Inductive Step:** We will attempt to ***prove*** $P(n+1)$, i.e

$$a_{n+1} = 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1}$$

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- Since $(n \geq 1), (n+1 \geq 2)$ and we can use the recursive definition of a.
- From the recursive definition of a we have:

$$a_{n+1} = 5a_n - 6a_{n-1} \overset{I.H}{=} 5(5 \cdot 3^n + 7 \cdot 2^n) - 6\left(5 \cdot 3^{n-1} + 7 \cdot 2^{n-1}\right)$$
$$= 25 \cdot 3^n + 35 \cdot 2^n - 30 \cdot 3^{n-1} - 42 \cdot 2^{n-1}$$
$$= 5 \cdot (5 \cdot 3^n - 2 \cdot 3^n) + 7(5 \cdot 2^n - 3 \cdot 2^n) = 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1} \quad \square$$

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$$= 25 \cdot 3^n + 35 \cdot 2^n - 30 \cdot 3^{n-1} - 42 \cdot 2^{n-1}$$
$$= \underbrace{5 \cdot (5 \cdot 3^n - 2 \cdot 3^n) + 7(5 \cdot 2^n - 3 \cdot 2^n)}_{\text{Since we need factors of 5 and 7 in our result, we force them to appear and our lives automatically become easier!}} = 5 \cdot 3^{n+1} + 7 \cdot 2^{n+1} \quad \square$$

A Sequence Problem for You!

- Let a_n be defined as:

$$a_n = \begin{cases} 5, & n = 0 \\ 16, & n = 1 \\ 7a_{n-1} - 10a_{n-2}, & n \geq 2 \end{cases}$$

- Prove that $a_n = 3 \cdot 2^n + 2 \cdot 5^n$
- Breakout Rooms

Another Sequence Problem

- Let a_n be defined as:

$$a_n = \begin{cases} 3, & n = 0 \\ 5, & n = 1 \\ 3a_{n-1} - 2a_{n-2}, & n \geq 2 \end{cases}$$

- Prove that $a_n = 2^{n+1} + 1$

Important Note

- In our proofs on recurrences, $P(n+1)$ dependent on stuff such as

$$P(n), P(n-1), P(n-2), \ldots$$

- It is possible (and common) for $P(n+1)$ to depend on

$$P\left({}^{(n+1)}/_{2}\right), P\left({}^{(n+1)}/_{3}\right), P(\sqrt{n+1}) \ldots$$

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Modified Strong Induction Principle

- BS: $P(a)$
- IS: $(\forall\, n \geq a)[P(a) \land \cdots\ P(n-1)\ \Rightarrow P(n)]$

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- IS: $(\forall\, n \geq a)[P(a) \wedge \cdots\ P(n-1)\ \Rightarrow P(n)]$
- From these two you can deduce
  - For all $n \geq a$, $P(n)$.
- Why does the Modified Strong Induction Principle Work?
  - Similar to who the original Strong Induction Principle worked.

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