An Interesting Sum
Examples

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\frac{1}{1} + \frac{1}{2} + \frac{1}{1 \times 2} = 2.
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Let's try it with \{1, 2, 3, 4\}. 
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Lets try it with \(\{1, 2, 3, 4\}\).

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\]
Let \( n \in \mathbb{N} \). Let \( S_n \) be the **multiset**

\[
S_n = \bigcup_{k=1}^{n} \{a_1 \cdots a_k : a_1, \ldots, a_k \in \{1, \ldots, n\} \land (\forall i, j)[a_i \neq a_j]\}.
\]

**Examples**

\[S_1 = \{1\}\]

\[S_2 = \{1, 2, 1 \times 2\} = \{1, 2, 2\}\]

\[S_3 = \{1, 2, 3, 1 \times 2, 1 \times 3, 2 \times 3, 1 \times 2 \times 3\} = \{1, 2, 3, 2, 3, 6, 6\}\]
Notation

Let \( n \in \mathbb{N} \). Let \( S_n \) be the multiset

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S_n = \bigcup_{k=1}^{n} \{ a_1 \cdots a_k : a_1, \ldots, a_k \in \{1, \ldots, n\} \wedge (\forall i, j)[a_i \neq a_j]\}.
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Restate and Add to Initial Examples

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\[ \sum_{x \in S_4} \frac{1}{x} = 4. \]
Goal

We will prove the following:

\[ \text{Theorem} \]

For all \( n \geq 1, \) \( X_x \in S_n \) \( x = n. \)

To prove this we first need a lemma.
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We will prove the following:

**Theorem** For all $n \geq 1$,

$$\sum_{x \in S_n} \frac{1}{x} = n.$$
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To prove this we first need a lemma.
Lemma For all $n \geq 2$,
(a) $S_n = S_{n-1} \cup n \cdot S_{n-1} \cup \{n\}$.
(b) $\sum_{x \in S_n} \frac{1}{x} = \sum_{x \in S_{n-1}} \frac{1}{x} + \sum_{x \in S_{n-1}} \frac{1}{nx} + \frac{1}{n}$. 
Lemma For all $n \geq 2$,

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Proof This is not by induction on $n.$
Lemma For all $n \geq 2$,
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Proof This is not by induction on $n$.
What is in $S_n$?
Lemma

For all $n \geq 2$,

(a) $S_n = S_{n-1} \cup n \cdot S_{n-1} \cup \{n\}$.

(b) $\frac{1}{x} = \sum_{x \in S_n} \frac{1}{x} = \sum_{x \in S_{n-1}} \frac{1}{x} + \sum_{x \in S_{n-1}} \frac{1}{nx} + \frac{1}{n}$.

Proof This is not by induction on $n$.

What is in $S_n$?

1. Everything from $S_{n-1}$.
Lemma

For all \( n \geq 2 \),
(a) \( S_n = S_{n-1} \cup n \cdot S_{n-1} \cup \{n\} \).
(b) \( \sum_{x \in S_n} \frac{1}{x} = \sum_{x \in S_{n-1}} \frac{1}{x} + \sum_{x \in S_{n-1}} \frac{1}{nx} + \frac{1}{n} \).

Proof This is not by induction on \( n \).

What is in \( S_n \)?

1. Everything from \( S_{n-1} \).
2. Prod. of \( n \) with a set of dist elts from \( \{1, \ldots, n-1\} \).
Lemma For all $n \geq 2$,

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What is in $S_n$?

1. Everything from $S_{n-1}$.

2. Prod. of $n$ with a set of dist. elts from $\{1, \ldots, n-1\}$.

   2.1 If the set of dist. elts is $\neq \emptyset$ then this is $n \cdot S_{n-1}$.
Lemma For all $n \geq 2$,

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What is in $S_n$?

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   2.1 If the set of dist. elts is $\neq \emptyset$ then this is $n \cdot S_{n-1}$.
   2.2 If the set of dist. elts is $\emptyset$ then this is $\{n\}$.
**Lemma**  For all $n \geq 2$,

(a) $S_n = S_{n-1} \cup n \cdot S_{n-1} \cup \{n\}$.

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What is in $S_n$?

1. Everything from $S_{n-1}$.
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   1. If the set of dist. elts is $\neq \emptyset$ then this is $n \cdot S_{n-1}$.
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Putting this all together you get the lemma.
Main Theorem

Theorem

For all \( n \geq 1 \), \( P \exists x \in S^n \), \( x = n \).

Proof

We prove this by induction on \( n \).

Base Case

\( S^1 = \{1\} \) so \( P \exists x \in S^1 \), \( x = 1 \).

Inductive Hypothesis

\( P \exists x \in S^{n-1} \), \( x = n-1 \).

Inductive Step

By the Lemma \( \exists x \in S^n \), \( x = x + 1 \).

By the IH this is \( n - 1 + 1 = n \).
Main Theorem

**Theorem** For all $n \geq 1$, $\sum_{x \in S_n} \frac{1}{x} = n$. 
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Base Case $S_1 = \{1\}$ so $\sum_{x \in S_1} \frac{1}{x} = 1$. 
**Theorem** For all \( n \geq 1 \), \( \sum_{x \in S_n} \frac{1}{x} = n \).

**Pf** We prove this by induction on \( n \).

**Base Case** \( S_1 = \{1\} \) so \( \sum_{x \in S_1} \frac{1}{x} = 1 \).

**IH** \( \sum_{x \in S_{n-1}} \frac{1}{x} = n - 1 \).
Main Theorem

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Main Theorem

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$$\sum_{x \in S_n} \frac{1}{x} = \sum_{x \in S_{n-1}} \frac{1}{x} + \sum_{x \in S_{n-1}} \frac{1}{nx} + \frac{1}{n}.$$
**Main Theorem**

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Main Theorem

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By the IH this is

$$n - 1 + \frac{1}{n} (n - 1) + \frac{1}{n} = n - 1 + n - \frac{1}{n} + \frac{1}{n} = n.$$