Duplicator Spoiler Games

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We will call SPOIL S and DUP D to fit on slides.

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Bill plays a student $(L_3, L_4, 2)$**,** $(L_3, L_4, 3)$

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3. GENERALLY: Who wins (L_a, L_b, k).

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Play a student $\mathbb{N}$ and $\mathbb{Z}$ with 1 move, 2 moves

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4. D wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k-1)$, S wins $(L_{10}, \mathbb{N} + \mathbb{N}^*, k)$.

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Let L and L' be two linear orderings.

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Let L and L' be two linear orderings.
Def If D wins the k-round DS-game on L, L' then L, L' are *k-game equivalent* (denoted $L \equiv_k^G L'$).

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If $Q \in \{\exists, \forall\}$ then

$$\mathrm{qd}((Qx_1)[\phi(x_1, \ldots, x_n)] = \mathrm{qd}(\phi_1(x_1, \ldots, x_n)) + 1.$$

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$$\mathrm{qd}((\forall x)(\forall z)[x < z \rightarrow (\exists y)[x < y < z]]) = 2 + 1 = 3$$

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Def L and L' are *k-truth-equiv* ($L \equiv_k^T L'$)

$$(\forall \phi, qd(\phi) \leq k)[L \models \phi \text{ iff } L' \models \phi.]$$

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4. Complexity: As Computer Scientists we think of complexity in terms of time or space (e.g., sorting n elements can be done in roughly $n \log n$ comparisons). But how do you measure complexity for concepts where time and space do not apply? One measure is quantifier depth. These games help us prove LOWER BOUNDS on quantifier depth!

Proving DUP Wins Rigorously

Notation

The game where the orders are L and L', and its for n moves, will be denoted

$$(L, L'; n)$$

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  By IH DUP wins $(L_{a-x}, L_{b-x}; n-1)$.

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3. Might not need induction on the smaller boards if they are orderings we already proved things about.

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$\mathbb{N}$ and $\mathbb{N} + \mathbb{Z}$

Thm For all n, DUP wins $(\mathbb{N}, \mathbb{N} + \mathbb{Z}; n)$.
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1) If SP 1st move is x in either $\mathbb{N}$ or $\mathbb{N}$-part of $\mathbb{N} + \mathbb{Z}$ then DUP counters with the same x in the other part. The 2 games are

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$(L_x, L_x; n-1)$ and $(\mathbb{N}, \mathbb{N}+\mathbb{Z}; n-1)$.
SP won't play on 1st board. The 2nd board *DUP* wins by IH

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2) If SP 1st move is x in $\mathbb{Z}$ part of $\mathbb{N}+\mathbb{Z}$ then DUP plays 2^n in $\mathbb{N}$.
The 2 games are $(\mathbb{N}+\mathbb{N}^*, L_{2^n}; n-1)$ and $(\mathbb{N}, \mathbb{N}; n-1)$.

SP won't play on 2nd board.
By last slide, on 2nd board DUP wins $(\mathbb{N}, \mathbb{N}; n-1)$.

$\mathbb{Z}$ and $\mathbb{Z}+\mathbb{Z}$

Thm For all n, DUP wins $(\mathbb{Z}, \mathbb{Z}+\mathbb{Z}; n)$.

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SP 1st move is x is $\mathbb{Z}$. DUP picks x in first copy of $\mathbb{Z}$ in $\mathbb{Z}+\mathbb{Z}$.

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LEAVE THE REST TO YOU.

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You only have to do the cases that SP picks $x \in Z$.