BILL AND EMILY RECORD LECTURE!!!!

Solving Recurrences

Solving Recurrences: Fib

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Better idea Lets Derive it.

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We will combine them and modify them to fit base case.

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Upshot The set of solutions to $a_n = a_{n-1} + a_{n-2}$ is closed under addition and scalar multiplication (its a vector space).

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Upshot For any constants c, d $c\alpha_1^n + d\alpha_2^n$ satisfy the recurrence.

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Upshot The recurrence is solved by

$$a_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}.$$

Solving Recurrences: Distinct Roots Case

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(1) They come up in problems in combinatorics.
(2) They model some real world phenomena like Population Growth or the spread of an infection.
(3) To solve Differential Equations sometimes they are made discrete and become difference equations.
(4) Note: In CMSC 351 you will look at equations like

$$a_n = 2a_{n/2} + n$$

Which are used to analyze algorithms. That is NOT todays topic.

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Upshot For any constants $c_1, \ldots, c_k$

$$c_1\alpha_1^n + \cdots + c_k\alpha_k^n$$

is a solution to the recurrence.

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For $0 \leq L \leq k-1$:
Use a_L:

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That can be solved.

Then you have the closed form solution.

Solving Recurrences: The Non-Distinct Roots Case

An Example

Recall the Bill Sequence:

$a_0 = 0$

$a_1 = 1$

$a_2 = 2$

$(\forall n \geq 2)[a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}]$.

Ignore The Base Case for a While

Plan For now find **all** solutions to
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α_1^n and α_2^n and α_3^n all satisfy the recurrence.

But we need **three** solutions to make all of this work.

This is So Crazy it Might Just Work

Lets see if $n2^n$ is a solution to just the recurrence.

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OH, that worked!

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Might not be distinct.

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Let $\alpha_1, \ldots, \alpha_k$ be the roots.

Might not be distinct.

If α_i appears L times then
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Why We will not to into this but it involves that any multiple root of $p(x)$ is also a root of $p'(x)$.

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We now have n **different** solutions which we will call:
$n^{j_1}\alpha_1^n$, $n^{j_2}\alpha_2^n$, ..., $n^{j_k}\alpha_k^n$,
(most of the j's are 0).
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Then you have the closed form solution.

More Complicated Recurrences

An Example

Recall the Emily Sequence:

$a_0 = 0$

$a_1 = 1$

$(\forall n \geq 2)[a_n = 5a_{n-1} - 6a_{n-2} + n^2]$.

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The proof is algebra which we will skip.

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Plan For now find **all** solutions to

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We will then add them.

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GOTO next slide

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Particular solution is

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For any constants c, d

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satisfy the recurrence.
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We leave this to the reader.

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If the extra term is BLAH then guess something of BLAH form with undetermined coefficients.

General Algorithm

We are not going to present the general algorithm for a general recurrence with an extra term since we are confident you an do that yourself.

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If you have a course in Diff Equations later then when you take it you will have a sense of Deja Vu, because of this course.

BILL AND EMILY STOP RECORDING LECTURE!!!!