Loaded Dice

Fair Dice Yield Unfair Sums

Fair Die:

$Pr(1)=Pr(2)=Pr(3)=Pr(4)=Pr(5)=Pr(6) = 1/6 \sim 0.167$

Roll TWO of them.

$Pr(Sum=2)=1/36$ (This is Min $Pr(Sum)$)
$Pr(Sum=7)=1/6$. (This is Max $Pr(Sum)$)

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How Unfair?: $1/6 - 1/36 \sim 0.139$ unfair.

What are Loaded Dice?

Def A **Die** is a 6-tuple $(p_1, p_2, p_3, p_4, p_5, p_6)$ such that $0 \leq p_i \leq 1$ and $\sum_{i=1}^{6} p_i = 1$.

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1) There exists a way to load dice so that all sums are prob $\frac{1}{11}$.
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1) There exists a way to load dice so that all sums are prob $\frac{1}{11}$.
2) There is no way to load dice so that all the sums are prob $\frac{1}{11}$.
No such dice can exist!

Polynomials are our Friends!

Assume that are dice that yield fair sums. Let $(p_1, \ldots, p_6)$ and $(q_1, \ldots, q_6)$ be those dice.
KEY:

$$(p_1 x + p_2 x^2 + \cdots + p_6 x^6)(q_1 x + q_2 x^2 + \cdots + q_6 x^6)$$

Coefficient of x^5 is

$$p_1 q_4 + p_2 q_3 + p_3 q_2 + p_4 q_1 = \mathrm{Prob}(\mathrm{sum} = 5)$$

Coefficient of x^i is $\mathrm{Prob}(\mathrm{sum} = \mathrm{i})$.

Fair Sums- NOT!

Let $(p_1, \ldots, p_6)$ and $(q_1, \ldots, q_6)$ be dice. **Assume** they yield FAIR SUMS, all sums have prob $1/11$. Then

$$(p_1 x + \cdots + p_6 x^6)(q_1 x + \cdots + q_6 x^6) = (1/11)(x^2 + x^3 + \cdots + x^{12})$$

So

$$(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5) = (1/11)(1 + x + x^2 + \cdots + x^{10})$$

Two Polynomials

Recap If $(p_1, \ldots, p_6)$ and $(q_1, \ldots, q_6)$ are two loaded dice that yield fair sums then:

$$(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5) = (1/11)(1 + x + x^2 + \cdots + x^{10})$$

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$$11(p_1 + \cdots + p_6 x^5)(q_1 + \cdots + q_6 x^5)(x - 1) = x^{11} - 1$$

Real Roots of Left Polynomials

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Upshot The Left poly has ≥ 3 real roots.

Real Roots of Right Polynomials (II)

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Recall Upshot The Left poly has ≥ 3 real roots.

Lets look at the roots of the right poly:
$x^{11} - 1 = 0$
$x^{11} = 1$
All roots on complex unit circle. Hence ≤ 2 real roots.
Upshot The Right poly has ≤ 2 real roots.

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$x^{11} = 1$
All roots on complex unit circle. Hence ≤ 2 real roots.
Upshot The Right poly has ≤ 2 real roots.
Final Upshot The left and right poly DIFFER on the number of real roots, so they cannot be the same. Contradiction!