$5^{1/3}$ is irrational

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Proof: For the sake of contradiction assume $5^{1/3}$ is rational. If $5^{1/3}$ is rational then,

$$5^{1/3} = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$ and $q \neq 0$ and there are no common factors between p and q.

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However, this means q^3 has to be divisible by 5. Hence we have a contradiction since we stated that p and q have no common factors. Therefore, $5^{1/3}$ is irrational. ☽

Prove $5^{1/3}$ is irrational

Proof: For the sake of contradiction assume that $5^{1/3} = \frac{a}{b}$. So

$$5 = \frac{a^3}{b^3}.$$

$$5b^3 = a^3.$$

Prove $5^{1/3}$ is irrational

Let $p_1, \ldots, p_L$ be all of the primes that divide either a or b. (We do not know or care if 5 is one of the p_i's.) Then by Unique factorization there is a unique $a_1, \ldots, a_L$ and $b_1, \ldots, b_L$ such that

$$a = p_1^{a_1} \cdots p_L^{a_L}$$

$$b = p_1^{b_1} \cdots p_L^{b_L}$$

Prove $5^{1/3}$ is irrational

$$a = p_1^{a_1} \cdots p_L^{a_L}$$

$$b = p_1^{b_1} \cdots p_L^{b_L}$$

So

$$5p_1^{3b_1} \cdots p_L^{3b_L} = p_1^{3a_1} \cdots p_L^{3a_L}.$$

Let LHS be the number of times 5 appears on the left. $LHS \equiv 1 \pmod 5$.
Let RHS be the number of times 5 appears on the right. $RHS \equiv 0 \pmod 5$.
Since $LHS = RHS$, we have a contradiction. ☽