Another Proof by Contradiction: The Set of Primes is Infinite

Infinity of primes

- Assume that the primes are finite. Then, we can list them in ascending order:

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Clearly, N is bigger than any p_i. We have two cases:

i. N is prime. Contradiction, since N is bigger than any prime.

ii. N is composite. This means that N has at least one factor f. Let's take the smallest factor of N, and call it f_{min}. Then, this number is prime (why?) Since f_{min} is prime, it divides $p_1 \cdot p_2 \cdot \ldots \cdot p_n$. By the previous theorem, this means that it cannot possibly divide $p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1 = N$. **Contradiction,** since we assumed that f_{min} is a factor of N.

Therefore, the primes are not finite.