**250 Untimed Midterm 2**
**Morally Due Monday April 8**

# 1 An Interesting Sum (7 points)

For this problem you may approximate

$$(n-1)^{11}$$

by

$$n^{11} - 11n^{10}$$

any time it appears.

Consider the sum

$$\sum_{i=100}^{n} i^{10}.$$

(**NOTE** that the summation starts at 100, so the base case will be $n = 100$.)

We are NOT going to ask you to get a closed form for it (there is one but its a mess). **BY CONSTRUCTIVE INDUCTION** find a constant $A$ such that

$$(\forall n \ge 100)\left[\sum_{i=100}^{n} i^{10} \le An^{11}\right].$$

Try to make $A$ as small as possible.

(**NOTE-** There are other ways to do this but I want a proof by

**CONSTRUCTIVE INDUCTION.**

DO NOT be that guy who does it a different way and claims that its right. That wastes my time and yours.)

(**NOTE-** Problem 2 implies Problem 1. DO NOT be that guy who just does Problem 2 and claims he did Problem 1. **JUST DO BOTH PROBLEMS**.)

## 2 A Generalization of an Interesting Sum (8 points)

In this problem you may approximate

$$(n-1)^a$$

by

$$n^a - an^{a-1}$$

any time it appears,
Let $a \in \mathsf{N}$ be such that $a \geq 11$.
Consider the sum

$$\sum_{i=100}^{n} i^a.$$

**BY CONSTRUCTIVE INDUCTION** find a constant $B$ such that

$$(\forall n \geq 100)\left[\sum_{i=100}^{n} i^a \leq Bn^{a+1}\right].$$

(**NOTE-** There are other ways to do this but I want a proof by
**CONSTRUCTIVE INDUCTION.**
DO NOT be that guy who does it a different way and claims that its right. That wastes my time and yours.)

## 3 A Coin Problem (15 points)

The Daleks only have two coins:

- a 10-cent coin, and
- a 13-cent coin.

Note that there are some amounts they cannot create. For example Davros cannot give Emily 22 cents. However, there is a $C \in \mathsf{N}$ such that

- $C - 1$ cannot be written as $10x + 13y$ where $x, y \in \mathsf{N}$, and
- $(\forall n \geq C)(\exists x, y \in \mathsf{N})[n = 10x + 13y]$.

And now finally the problem which will guide you to finding $C$.

1. Write a program that will, given $N \in \mathsf{N}$,

   determine, for all $1 \leq n \leq N$ if $n$ can be written as the sum of 10's and 13's.
2. Run the program on $N = 1000$.
3. Based on your numbers make a conjecture for what $C$ is.
4. Prove your conjecture by induction.

# 4 Sums of Squares (20 points)

In this problem we guide you through a proof that number of the form $16^n(16k+15)$ cannot be written as the sum of 14 fourth-powers.

1. (0 points but you will need it later) Find the following set

$$X = \{x^4 \pmod{16} : x \in \{0, \ldots, 15\}\}.$$

2. (3 points) Show that if $x \equiv 15 \pmod{16}$ then $x$ is NOT the sum of 14 fourth-powers.

   (*Hint:* Show that any 14 elements of $X$ from Part a can never sum to 15 mod 16.)

3. (3 points) Show that if $x$ is odd then $x^4 \equiv 1 \pmod{16}$.

   **Hints**

   - Begin the proof by letting $x = 2k+1$.
   - Look up the Binomial Theorem. You will need it.
   - Note that, for all $k \in \mathsf{N}$, $k(k+1)$ is even. You will need this fact.

4. (4 points) Prove that if $x_1^4 + \cdots + x_{14}^4 \equiv 0 \pmod{16}$ then

$$(\forall i)[x_i \equiv 0 \pmod{2}].$$

5. (10 points) Prove the following by induction on $n$.

   **Theorem** Let $n \geq 0$. Let $k \in \mathsf{N}$. Show that $16^n(16k+15)$ cannot be written as the sum of 14 fourth-powers.